International Journal of Science and Engineering (IJSE)

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Simulation of Traffic Flow Model with Traffic Controller Boundary

Nahid Sultana[1], Masuma Parvin[2], Ronobir Sarker[3], Laek Sazzad Andallah[4]

[1]Bangladesh University, Dhaka, Bangladesh, Email: nahid.nipu1@yahoo.com
[2] Daffodil International University, Dhaka, Bangladesh, E-mail: laizu416@yahoo.com
[3]Bangladesh University of Business and Technology, Dhaka, Bangladesh, E-mail: ronobir.sarker@gmail.com
[4]Jahangirnagar University, Savar, Dhaka, Bangladesh, E-mail: andallahls@gmail.com

ABSTRACT- This paper considers a fluid dynamic traffic flow model appended with a closure linear velocity-density relationship which provides a first order hyperbolic partial differential equation (PDE) and is treated as an initial boundary value problem (IBVP). We consider the boundary value in such a way that one side of highway treat like there is a traffic controller at that point. We present the analytic solution of the traffic flow model as a Cauchy problem. A numerical simulation of the traffic flow model (IBVP) is performed based on a finite difference scheme for the model with two sided boundary conditions and a suitable numerical scheme for this is the Lax-Friedrichs scheme. Solution figure from our scheme indicates a desired result that amplitude and frequency of cars density and velocity reduces as time grows. Also at traffic controller point, velocity and density values change as desired manner. In further, we also want to introduce anisotropic behavior of cars(to get more realistic picture) which has not been considered here.

Keywords - tensity function; tinite difference scheme; macroscopic traffic flow model; nonlinear velocity; tumerical simulation.

Submission: March 12, 2013 | Corrected : June 20, 2013 | Accepted: June 23, 2013

Doi: 10.12777/ijse.x.x.xx-xx

[**How to cite this article**: Sultana, N., Parvin, M. , Sarker, R., Andallah, L.S. (2013). Simulation of Traffic Flow Model with Traffic Controller Boundary. *International Journal of Science and Engineering*, 5(1),6-11. Doi: 10.12777/ijse.x.x.xx-xx]

I. INTRODUCTION

At present time we cannot deal our life not a single day without vehicles likes bus, car, taxi, rickshaw etc. for our communication. So traffic flow and congestion is related to our transportation. At this time traffic congestion is one of the vital problems in our country like other developing countries of the globe.

In mathematics traffic flow is the study of interactions between vehicles, drivers, and infrastructure (including highways, signage, and traffic control devices), with the aim of understanding and developing an optimal road network with efficient movement of traffic and minimal traffic congestion problems.

Traffic Phenomena are complex and nonlinear, depending on the interactions of a large number of vehicles. Due to the individual reactions of human drivers, vehicles do not interact simply following the laws of mechanics, but rather show phenomena of cluster formation and shock wave propagation, both forward and backward, depending on vehicle density in a given area.

In the fluid flow analogy, the traffic stream is treated as a one dimensional compressible flow. This leads to two basic assumptions :(a) traffic flow is conserved and b) there is a one-to-one relationship between speed and density or between flow and density. As mentioned by (Daganzo1995), the difference between traffic and fluid as follows: A fluid particle responds to the stimulus from the front and from behind, however a vehicle is an anisotropic particle that mostly responds to frontal stimulus. Many research groups are involved in dealing with the problem with different kinds of traffic models like the Microscopic car following model, the macroscopic fluid dynamic model and The Mesoscopic (Kinetic) model. All models describe various situations with different assumptions and simplifications. In this paper, we attention on macroscopic fluid dynamics

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model because it is more efficient and easy to implement than other modeling approaches.

The macroscopic traffic model was first developed by Lighthill and Whitham (1955) and Richard (1956). This model is also shortly called LWR. According this model, vehicles in traffic flow are considered as particles in fluid; further the behavior of traffic flow is described by the method of fluid dynamics and formulated by hyperbolic partial differential equation. This model is used to study traffic flow by collective variables such as traffic flow rate (flux) $q(x,t)$, traffic speed $v(x,t)$ and traffic density $\rho(x,t)$, all of which are functions of space, $x \in R$ and time $t \in R^+$. The most well-known LWR model is formulated by the conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0$$

The problem of computer simulation techniques of the traffic flow models has become an interesting area in the field of numerical solution methods for several decades. For instance, in (Daganzo1995) the author shows that if the kinematics wave model of freeway traffic flow in its general form is approximated by a particular type of finite difference equation, the finite difference results converge to the kinematics wave solution despite the existence of shocks in the latter. Errors are shown to be approximately proportional to the mesh spacing with a co-efficient of proportionality that depends on the wave speed, on its rate of change with density and on the slope and curvature of the initial density profile. The asymptotic errors are smaller than those of Lax's first order, centered difference method which is also convergent. In the paper (Zhang 2001) the author develops a finite difference scheme for non-equilibrium traffic flow model.

This scheme is an extension of Godunov's scheme (Randall .J .Leveque, 1992) to system. It utilizes the solutions of a series of Riemann problems at cell boundaries to construct approximate solutions of the non-equilibrium traffic flow model under general initial conditions. Moreover, the Riemann solutions at both left (upstream) and right (downstream) boundaries of a highway allow the specification of correct boundary conditions using state variables (e.g. density and / or speed) rather than fluxes. Preliminary numerical results indicate that the finite difference scheme correctly computes entropy-satisfying weak solutions of the original model. In Bretti *et al.* (2007) the authors consider a mathematical model for fluid dynamic flows on networks which is based on conservation laws. The approximation of scalar conservation laws along arcs is carried out by using conservative methods, such as the classical Godunov scheme and the more recent discrete velocities kinetic schemes with the use of suitable boundary conditions at junctions. Riemann problems are solved by means of a simulation algorithm which processes each junction. In the paper (Haberman, 1977 and Klar *et al.*, 1996) authors consider the LWR model. This model describes traffic phenomena resulting from interaction of many vehicles by discussing the fundamental traffic variables like density, velocity and flow. In particular, a linear velocity-density relationship which yields a quadratic flux-density relation yields a formulation of traffic problem as a first order non-linear partial differential equation. The exact solution of the nonlinear PDE as a Cauchy problem is presented. Moreover, in order to incorporate initial and boundary data, the non-linear first order (PDE) is appended by initial and boundary value which leads to formulate an initial boundary value problem (IBVP).Certainly, a numerical method is needed for the numerical implementation of the IBVP in practical situation and it is completely unavoidable to use numerical method to solve real traffic problem. A numerical scheme for the model which is applicable for any velocity density model including discontinuous one is presented in (Kuhne *et al.*, 1997). But in this scheme two sided boundary condition is needed and the accuracy of the scheme is not satisfactory. However, there is a requirement of using one sided boundary condition. In (Andallah *et al.*, 2009) author considers explicit upwind difference scheme for traffic model where he considers one sided boundary condition. In this paper first, we present the derivation of the macroscopic traffic flow model with corresponding variables relationship among velocity, flux and density. Then, we present the analytic solution of the traffic flow model with linear and nonlinear partial differential equation (PDE) by the method of characteristics and the existence and uniqueness of the analytic solution of the traffic flow model. We develop a finite difference scheme for our traffic flow model as an (IBVP) which has been presented before numerical simulation section. We develop computer programming code for the implementation of the numerical scheme and perform numerical experiments in order to verify some qualitative traffic flow behavior for various traffic parameters. Finally, we present the numerical simulation.

II. MACROSCOPIC TRAFFIC FLOW MODEL

We present an outline of mathematical modeling, a partial differential equation as a mathematical model for traffic flow. Also we represent specific speed-density relationship and specific flow-density relationship and the well-known LWR model (Haberman, 1977 and Klar *et al.*, 1996) based on the principle of mass conservation. We assume all cars have some constant velocity $v > 0$. Then from the relationship among velocity, flux and density, the flux $q = \rho v$ yields the equation of continuity $\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$ in the form

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0 \qquad (1)$$

a linear first order partial differential equation (PDE) called linear advection equation, which in closed form i.e.

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solvable. In this case we consider velocity $v = v(\rho)$ **(2)**
as a function of density then we have the equation (1) as the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v(\rho))}{\partial x} = 0 \qquad \textbf{(3)}$$

Which is a first order PDE and nonlinear in ρ .This yield

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{dv}{d\rho}\frac{\partial \rho}{\partial x} = 0$$
$$\Rightarrow \frac{\partial \rho}{\partial t} + \left(v + \rho\frac{dv}{d\rho}\right)\frac{\partial \rho}{\partial x} = 0 \qquad \textbf{(4)}$$

which is linear in derivatives but non-linear in ρ , termed as quasi-linear equation. We use a non-linear speed-density relation (non-linear function) which is of the form

$$v(\rho) = v_{\max}\left(1 - \left(\frac{\rho}{\rho_{\max}}\right)^2\right) \qquad \textbf{(5)}$$

with the aid of equation (5) give a relationship for the traffic flux (flow) as a function of density that is presented in the following form:

$$q(\rho) = v_{\max}\left(\rho - \frac{\rho^3}{\rho_{\max}^2}\right) \qquad \textbf{(6)}$$

Now if we put the velocity-density function $v(\rho) = v_{\max}\left(1 - \left(\frac{\rho}{\rho_{\max}}\right)^2\right)$ of equation (5) into the general non-linear model PDE (4), then the explicit non-linear PDE is obtained as in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho . v_{\max}\left(1 - \frac{\rho^2}{\rho_{\max}^2}\right)\right) = 0$$
$$\text{With } \rho(t_0, x) = \rho_0(x) \qquad \textbf{(7)}$$

Then we are required to solve the initial value problem (IVP) (7). The IVP (7) can be solved by the method of characteristics as follows: The PDE in the IVP (7) may be writtenas, $\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0$ where $q(\rho) = \rho . v_{\max}\left(1 - \frac{\rho^2}{\rho_{\max}^2}\right)$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = 0$$
$$\Rightarrow \frac{\partial \rho}{\partial t} + v_{\max}\left(1 - \frac{3\rho^2}{\rho_{\max}^2}\right)\frac{\partial \rho}{\partial x} = 0$$
$$\text{Now} \quad \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} = 0$$

By integrating $\int d\rho = 0 \qquad \rho(t, x) = cons\tan t$
(8)

$$\text{Where} \quad \frac{dx}{dt} = v_{\max}\left(1 - \frac{3\rho^2}{\rho_{\max}^2}\right) \qquad \textbf{(9)}$$

Equation (8) and (9) gives $x(t) = v_{\max}\left(1 - \frac{3\rho^2}{\rho_{\max}^2}\right)t + x^0$
(10)
Which are the characteristics of the IVP (7).Now from equation (8) we have

$$\frac{d\rho}{dt} = 0 \quad \therefore \rho(x,t) = c \qquad \textbf{(11)}$$

Since the characteristics through (x,t) also passes through $(x^0, 0)$ and $\rho(x,t) = c$ is constant on this curve, so we use the initial condition to write

$$c = \rho(x,t) = \rho(x^0, 0) = \rho_0(x^0) \qquad \textbf{(12)}$$

Equation (11) and (12) yield

$$\rho(x,t) = \rho_0(x^0) \qquad \textbf{(13)}$$

Using equation (10), (13) takes the form

$\rho(x,t) = \rho_0\left(x - v_{\max}\left(1 - \frac{3\rho^2}{\rho_{\max}^2}\right)t\right)$ This is the analytic solution of the IVP (7).

III. A FINITE DIFFERENCE SCHEME FOR THE NUMERICAL SOLUTION OF THE IBVP

We investigate a finite difference scheme for our considered traffic model problem as an initial and two point boundary value problems

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial f(\rho)}{\partial x} = 0, \ t_0 \le t \le T, \ a \le x \le b \\ \text{with I.C.} \quad \rho(t_0, x) = \rho_0(x); \ a \le x \le b \\ \text{and B.C.} \quad \rho(\text{t, a}) = \rho_{\text{a}}(t); \ t_0 \le t \le T \\ \qquad \rho(\text{t, b}) = \rho_{\text{b}}(t); \ t_0 \le t \le T \end{cases}$$

Where

$$f(\rho) = \rho . v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right) \qquad \textbf{(14)}$$

As we consider that the cars are running only in the positive x-direction, so the speed must be positive,

i.e. $f'(\rho) = v_{\max}(1 - \frac{2\rho}{\rho_{\max}}) \ge 0$

in the range of ρ . **(15)**

To develop this scheme, we discretize the space and time. We discretize the time derivative $\frac{\partial \rho}{\partial t}$ in the IBVP (14) at any discrete point (t_n, x_i) for $i = 1, \cdots, M; n = 0, \cdots, N-1;$ by the forward

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difference formula. The discretization of $\frac{\partial \rho(x,t)}{\partial t}$ is obtained by first order forward difference in time and the discretization of $\frac{\partial \rho(x,t)}{\partial x}$ is obtained by first order central difference in space.

A. Forward Difference in Time:

From Taylor's series we write

$$\rho(x,t+k)=\rho(x,t)+k\frac{\partial \rho(x,t)}{\partial t}+\frac{k^2}{2!}\frac{\partial^2 \rho(x,t)}{\partial t^2}+\ldots$$

$$\Rightarrow \frac{\partial \rho(x,t)}{\partial t}=\frac{\rho(x,t+k)-\rho(x,t)}{k}-o(k)$$

$$\therefore \frac{\partial \rho(x,t)}{\partial t}\approx\frac{\rho(x,t+k)-\rho(x,t)}{k}$$

$$\therefore \frac{\partial \rho(t_n,x_i)}{\partial t}\approx\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t} \qquad \textbf{(16)}$$

B. Central Difference in Space:

From Taylor's series we write

$$f(x+h,t)=f(x,t)+h\frac{\partial f(x,t)}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 f(x,t)}{\partial x^2}+\ldots$$

$$f(x-h,t)=f(x,t)-h\frac{\partial f(x,t)}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 f(x,t)}{\partial x^2}-\ldots$$

$$\Rightarrow \frac{\partial f(x,t)}{\partial x}\approx\frac{f(x+h,t)-f(x-h,t)}{2h}$$

$$\Rightarrow \frac{\partial f(t_n,x_i)}{\partial x}\approx\frac{f_{i+1}^n-f_{i-1}^n}{2\Delta x} \qquad \textbf{(17)}$$

We assume the uniform grid spacing with step size k and h for time and space respectively $t^{n+1}=t^n+k$ and $x_{i+1}=x_i+h$. We also write ρ_i^n for $\rho(x,t)$.

Putting (16) and (17) in (14), the discrete version of the non-linear PDE formulates the first order finite difference scheme of the form

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}+\frac{f_{i+1}^n-f_{i-1}^n}{2\Delta x}=0$$

$$\rho_i^{n+1}=\rho_i^n-\frac{\Delta t}{2\Delta x}\left[f(\rho_{i+1}^n)-f(\rho_{i-1}^n)\right],$$

$$i=1,\cdots,M;\, n=0,\cdots,N-1 \qquad \textbf{(18)}$$

This difference equation is known as Lax-Friedrichs scheme.

Where $f(\rho_i^n)=v_{\max}\left(\rho_i^n-\frac{(\rho_i^n)^2}{\rho_{\max}}\right)$

IV. NUMERICAL SIMULATION

In this section we present the numerical simulation results for some specific cases of flow parameters like $\rho_{\max}$, $v_{\max}$ etc. We choose maximum velocity, v_{max} 84 km/hour. It is notified that for satisfying the CFL condition we pick the unit of velocity as km/sec. We consider ρ_{max} 40 ve□icles/ m and perform numerical experiment for 1, 2, 3 and 4 minutes in $\Delta t = 2400$ time steps (temporal grid size) for a highway of 120 km in 401 spatial grid points with step sizes $\Delta x = 100$ meters. We consider the initial density $\rho(0,x)$ and the constant boundary value ρ ,0 21/0.1 and ρ ,100 in such a way that rightmost side of traffic highway behaves as there is a traffic controller at right side .So we define ρ ,100 as follows:

$$\rho \quad \begin{matrix} 0\,;\, m & ,120 \;\le\; 60 \\ \rho_{max}\,;\, m & ,120 \quad 60 \end{matrix}$$

Figure 1, shows the propagating traffic waves at 1st, 2nd, 3rd and 4thminute's.we observe from the figure (1) that at 1 min, the density at right most side becomes zero at 2 and 3 minutes. Density becomes high at 4 minutes and average at 1 minute.

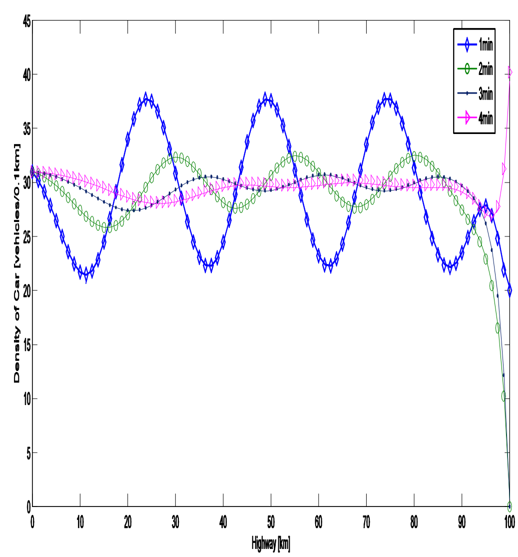

Figure 1: density of traffic flow at different times.

In the figure 1, the curve manifested as blue color, represents the density at 1 minutes .also manifested as green color represents density at 2 minutes and manifested as black color represents density at 3 minutes, also manifested as magenta color represents density at 4 minutes. Figure 2, represents the respective Computed velocity profile according to the certain Points of the highway. The velocity is computed by the following relation

$$v(\rho)=v_{\max}\left(1-\left(\frac{\rho}{\rho_{\max}}\right)^2\right)$$

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We consider the boundary condition in such a way that front of traffic highway behaves like a traffic controller. The right boundary is given as

_ _ 0.

;

; , $0 \leq 60$

0; ; , 0 60

For first 60 seconds, it behaves like that there is no restriction on traffic flow and we observe from the figure that at 1 min, the velocity at right most side of highway is high. But after 1 min, traffic controller restricts the traffic flow, and so velocity of traffic flow at right most side becomes very small. (From the figure it may seems that velocity is not too small at rightmost point of highway. This is due to two reasons. Firstly since the average velocity of cars is high so at small neighborhood of rightmost part we observe that rate of change of velocity is very high.

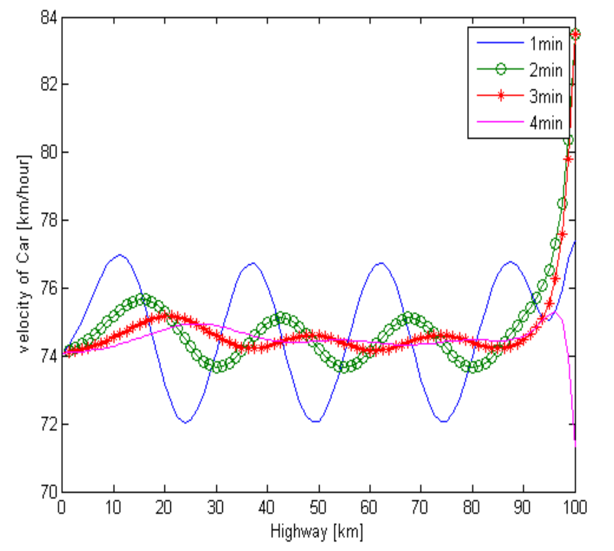
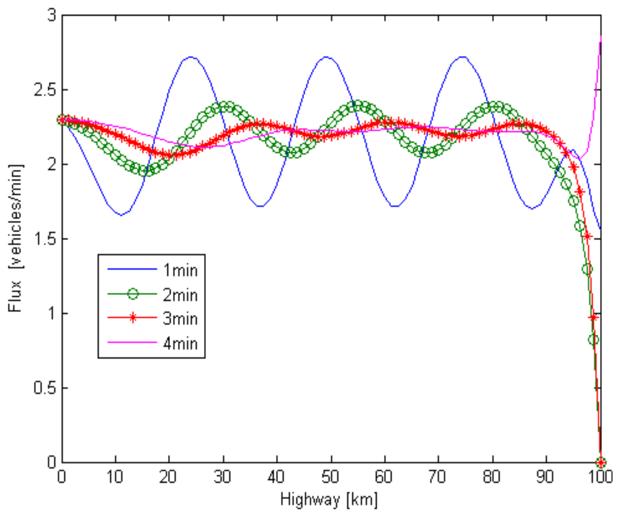


Figure 2: velocity profiles



Secondly if we consider increase the numerical grid points along the highway, then we should observe that velocity at rightmost point of highway is zero.) This restriction on velocity occurs periodically which is clear from the figure.

From the figure 3, we observe that at time t=1 min, Density is approximately 1.5 vehicles per time. But at time t=2 min, when we restrict car flow at the end point of highway, the flux is zero. It is obvious because at that time no car will pass through the end point and so the rate

of number of passing car will be zero. At time t=3, there is no restriction and so all cars begin to move and flux rises to approx. 3.We observe that as time grows fluctuation of flux decreases and flux get its extreme value at the end point of considered highway.

V. CONCLUSION

In this work we have considered the macroscopic traffic flow model. First we have shown the fundamental of traffic variables, analytical solution of the traffic flow model. We present the shock waves of traffic flow. We consider our specific non-linear traffic model problem as an initial and two point boundary value problems and use a suitable numerical scheme for this that is the Lax-Friedrichs scheme. Finally we show the numerical results for flux, density and velocity putting boundary value as flux. We have also chosen right boundary condition in such a way that it behaves like a traffic controller. We have considered a linear relationship between density and velocity but in practice this is not realistic. Firstly density function is not continuous in practice. Secondly if the next car's acceleration increases then the previous car would accelerate its velocity despite of higher density. So velocity at a point would depend on right limit of density, velocity and rate of change of velocity. We would work on this relationship in further.

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