MATHEMATICAL APPROACH OF RIVERBED PROPAGATION

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ABSTRAK

Berbagai macam pengambilan keputusan ideal yang berkaitan dengan rekayasa persungaian mensyaratkan sebuah studi menyeluruh pada perilaku sungai (respon sungai) baik akibat perubahan alam maupun akibat intervensi manusia. Dalam mempelajari respon sungai, terdapat banyak aspek yang harus diperhatikan, terutama yang berhubungan dengan morfologi sungai. Propagasi dasar sungai akibat perubahan lokal adalah salah satu aspek penting yang membutuhkan perhatian khusus karena dampaknya terhadap perubahan morfologi sungai. Sayangnya aspek ini belum dikaji dengan jelas karena fenomenanya yang kompleks. Tujuan utama tulisan ini adalah untuk mempelajari sekaligus mendapatkan fomulasi matematis bagaimana dasar sungai berperilaku akibat perubahan debit atau kekasaran bantaranya. Sebuah pendekatan matematis satu dimensi telah dirumuskan pada tulisan ini untuk menjelaskan secara sederhana proses konveksi - difusi yang terjadi pada propagaasi dasar sungai. Prinsip conveksi - difusi sebuah besaran dijadikan dasar pengembangan formula matematis dalam tulisan ini dengan mengadaptasikan batasan-batasanya dalam batasan-batasan morfologi sungai. Hasil dari penelitian ini menunjukkan bahwa pendekatan perilaku dasar sungai dengan metode matematis sederhana hanya dapat digunakan sebagai perkiraan umum. Hal tersebut dikarenakan kedua variabel: wave celerity (c_3) dan diffusion factor (D) telah diasumsikan sebagai variabel yang linear. Sedangkan pada kenyataanya, kedua variabel tersebut berperilaku tidak linear. Dengan demikian pemahaman atas kemungkinan pemberlakuan formula yang dihasilkan serta batasan-batasan berlakunya formula tersebut menjadi hal yang sangat penting diperhatikan. Pendekatan numeris sangat direkomendasikan pada penelitian lanjutan untuk mempelajari lebih lanjut perilaku propagasi dasar sungai.

***Kata kunci** : morfologi sungai, dasar sungai, pendekatan matemtis, konveksi-difusi*.

GENERAL

Mathematical approach is a very helpful tool to arrive at general understanding of such physical phenomena in nature. It is also very useful tool for a quick evaluation of the numerical solution based on the full mathematical equations. For the derivation of such analytical model, simplifications and linear assumptions have to be introduced. Therefore, awareness of the underlying assumptions and limitations is required to have a better understanding of the results obtained.

One of the most important physical phenomena in river morphology is river response due to local hydraulic changes. The response become very

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important since such changes in the upstream side affects the downstream side of the river. Based on studies done by Van der Klis (2000), Rath (2001) and Wahono (2002), it was founded that local hydraulic due to floodplain changes will initiates kind of local river bed disturbances which then propagate downstream and behaves as a wave.

In this research, mathematical equations were derived in order to explain the phenomenon of river bed-wave propagation. For the one-dimensional approach, the convection-diffusion equation was proposed, which has also been widely used in physics and water quality modelling.

Diffusion is well known as an irreversible process and it is responsible for energy dissipation within the fluid. Under one-dimensional conditions, transversal diffusion of a property can be described by the reduction of the magnitude of the property and at the same time increasing its distance boundary in such a way that the integral of the property is always the same. Convection is a process, which is responsible for travelling process of such property. Due to the convection process, the property will travel in the same direction as direction of the fluid's celerity. There are two very importance variables in describing the convection-diffusion process, notably the celerity and the diffusion factor. In case of the riverbed-wave phenomena, celerity of the riverbed (c_3) and D, as function of (h, s, u), represent the celerity and the diffusion process respectively.

Analytical models have been used to study river morphology since the original work done by De Vries (1985) for describing a morphological time-scale for aggradation and degradation of riverbed. Meanwhile, in this research, equations were derived to explain riverbed-wave propagation due to local hydraulic changes. One question then raised, is it possible to explain the wave behaviour using analytical approach? This study was then initiated to answer the question.

RESEARCH METHOD

In this research, new mathematical formulas were developed in order to explain physical behaviour of the riverbed response. Based on previous studies (van der Klis, 2000; Rath,2001; and Wahono,2002), the response behaves as a confection-diffusion wave. Therefore new equations were derived based on principle of convection-diffusion of mass.

Analytical study regarding convection-diffusion process on water quality problem was presented by Fischer (1968), Thomann (1973) and followed by Graf (1998), meanwhile application on morphological time-scale was done by De Vries (1985). On the numerical point of view, Vreugdenhil (1982) studied on numerical diffusion of the wave computation. Derivation of the new wave equation with boundaries of riverbed morphology was proposed in this study. The equations were then used to simulate the travelling as well as dampening process of the riverbed waves.

In order to arrive on a proper conclusion, five stages were used in this study:

1. Problem Identification and literatures study.
2. Providing proper basic equations
3. Setting boundary conditions.
   The governing variables were averaged over the cross section of the river. Therefore, it is possible to

take the basic equations for the unit width of the river. Moreover, it is mostly assumed that the sediment transport is governed by the local hydraulic conditions.
4. Derivation of one-dimensional equations for morphological model. The equations were derived based on four basic equations of motion.
5. Result of those stages will then be compared with result from numerical approach, presented by Rath (2001) for river Waal.

General solution developed will also be used to explain the basic behaviour of the wave.

RESULT AND DISCUSSIONS

Basic Equations

Analytically, the one-dimensional model of the riverbed morphological model can be studied using the propagation phenomena of a small disturbance at the riverbed. Here, some limitations have to be introduced and only problem with one space dimension is considered. This assumption implies that all characteristics of the river may be only a function of the distance (x) and time (t). As only one-space ordinate is considered, the basic equations will also be written for one-dimensional flow. There are four basic equations, which consist of two equations of water movement and two equations of sediment movement:

The two equations of water movement are:

Momentum equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} + g\frac{\partial z}{\partial x} = -g\frac{u|u|}{C^2 h} \quad \text{........(1)}$$

Continuity equation:

$$\frac{\partial a}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0 \quad \text{...................... (2)}$$

Where:
h = water depth,
u = water flow velocity and
z = bed level.
C = Chezy coefficient

Meanwhile, the two sediment equations are:

Sediment continuity:

$$\frac{\partial z}{\partial t} + \frac{\partial s}{\partial x} = 0 \quad \text{..................................... (3)}$$

And
Sediment transport formula (predictor):

s = f (u, other parameters) .... (4)

Assumed that except velocity (u), all of other parameters are constant during the propagation, therefore the equation (4) can be derived along the distance direction as:

$$\frac{\partial s}{\partial x} = \frac{df(u)}{du}\frac{\partial u}{\partial x} \quad \text{................................ (5)}$$

by eliminated $\partial s / \partial x$ from equation (3) and (5) there will be three differential equation for the remaining three dependent variables (velocity u(x,t), bed level z(x,t) and water depth h(x,t)).

As a result, a system of six partial differential equations is obtained. Those equations can be formulated within a 6x6 metric system as shown in equation (5a).

$$\begin{bmatrix} 1 & u & 0 & g & 0 & g \\ 0 & h & 1 & u & 0 & 0 \\ 0 & df/du & 0 & 0 & 1 & 0 \\ dt & dx & 0 & 0 & 0 & 0 \\ 0 & 0 & dt & dx & 0 & 0 \\ 0 & 0 & 0 & 0 & dt & dx \end{bmatrix} \begin{bmatrix} \partial u/\partial t \\ \partial u/\partial x \\ \partial h/\partial t \\ \partial h/\partial x \\ \partial z/\partial t \\ \partial z/\partial x \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \\ du \\ dh \\ dz \end{bmatrix} \quad ..(5a)]$$

The determinant of the matrix mush vanishes to allow the possibility of a discontinuous partial derivative. With $dt \neq 0$ and $c = dx/dt$ this lead to the relation as presenting in equation (6): (De Vries, 1985):

-c^3 + 2uc^2 + (gh – u^2 + g d*f*/du)c – ug d*f*(u)/du = 0 ....(6)

the equation (6) can be modified to a relationship between the following three dimensionless parameter :

relative celerity = $\Phi = c/u$
Froude number = Fr = u/sqrt(gh)
A transport parameter = ψ = h^{-1}*df*(*u*)/*du*

The transport parameter can be clarified by supposing that s = au^n where a and n contain all the parameters except u (water velocity) which govern the transport process. This assumption lead to:

$\psi = n\dfrac{s}{q}$ and ....(7)

The equation of (7) shows that ψ is proportional to the ratio of the transport of sand and water. This ratio is usually as small, in order of 10^{-5}. With substitutions of the dimensionless parameters into equation [6] then:

$$\Phi^3 - 2\Phi^2 + (1 - F^{-2} - \psi F^{-2})\Phi + \psi F^{-2} = 0$$
....(8)

Refer to equation [8], there are three roots of Φ_i. Two roots ($\Phi_{1,2}$) express the propagation of a disturbance at water surface. The third root Φ_3 express the propagation of a small disturbance at the riverbed.

De Vies (1985) presented equation [8] in graphical view represents dimensionless form. The graphic shows correlation between relative celerity Φ_3 = (c/u) and Froude number (F). It also presents that for sub-critical flow condition, F < 1, and for Φ_3 about 10^{-5}, the relative celerity of river bed will be in order of 10^{-5}, meanwhile for the two other roots ($\Phi_{1,2}$) have relative celerity in order of 1-10. For this reason, in this study, equations of riverbed wave were derived with assumption of a quasi-steady water flow condition.

Regarding equation (8) and the dimensionless correlation, some remarks can be drawn as following:

1. $\Phi_3 \cong c_3$ is propagation celerity of bed wave
2. $\Phi_{1,2} \cong c_{1,2}$ are the propagation celerity of the water surface wave
3. If $\Phi_3 \cong c_3$ = 0 it means fixed bed channel, and $\Phi_{1,2} = c_{1,2}$ = $u \pm \sqrt{gh}$
4. In the vicinity of critical flow condition, F =1, the smallest celerity for disturbance at the water level is influenced by the sediment transport. Therefore computations are influenced by $\Phi_3 = c_3$
5. For moderate Froude number (F< 0.6), It can be drawn that $|\Phi_{1,2}|$ >>> Φ_3, thus $\Phi_3 = \dfrac{\psi}{1-F^2}$, which

means ψ is not necessary = Φ_3 unless F <<<< 1.

For Quasi-steady water movement, the two water-celerities, $\Phi_{1,2}$, are much higher than the riverbed celerity.

Therefore in the further study only celerity of the riverbed (Φ_3) is considered most of the time. The equation of the morphodynamic can then be derived as presented in the following equations:

With $\Phi_{1,2}$ >>> Φ_3 then:

$$\frac{\partial u}{\partial x}\left(u-\frac{gq}{u^2}\right)+g\frac{\partial z}{\partial x}=R \text{ (Resistant term)} \quad (9)$$

with equation (4) and (6):

$$\frac{df(u)}{du}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial t}=0 \quad (10)$$

by eliminating $\frac{\partial u}{\partial x}$ the equation (10) become:

$$\frac{\partial z}{\partial t}+\left[\frac{gdf(u)/du}{(gq/u^2)-u}\right]\frac{\partial z}{\partial x}=R\frac{df(u)/dx}{(gq/u^2)-u} \quad (11)$$

the equation (11) can be translated as the following form:

$$\frac{\partial z}{\partial t}+c_3\frac{\partial z}{\partial x}=F(u) \quad (12)$$

in which:

$$c_3=\left[\frac{gdf(u)/du}{(gq/u^2)-u}\right] \quad (12a)$$

$$c_3=u\frac{\psi}{1-F^2} \quad (13)$$

Equation [12] is well known as the simple wave equation. The right-hand term of the equation allows for dampening process of the peak waves along the distance when it is travelling. The dampening process is function of flow velocity, meanwhile the value responsible for the diffusion of the wave's peak called diffusion coefficient. Diffusion process has been widely developed in heat transfer phenomena as well as in water quality modelling. In case of the riverbed wave, Vreugdenhil (1982) stated that diffusion coefficient can be defined as:

$$D=\frac{1}{3}C^2hs'(u,..).\frac{1}{u} \quad (14)$$

Implementing the derivation of s, the bulk volume of sediment, the equation (14) become:

$$D=\frac{1}{3}C^2h\left[\frac{0.05}{(1-\varepsilon)\sqrt{g}C^3\Delta^2{}_dD_{50}}5(u)^4\right].\frac{1}{u} \quad (15)$$

which:

$$s=\left[\frac{0.05}{(1-\varepsilon)\sqrt{g}C^3\Delta^2{}_dD_{50}}(u)^5\right] \quad (16)$$

(Engelund-Hansen formula)

Convection-Diffusion

Phenomena of convection-diffusion process actually occur in many physical processes, which contain transferring mass, energy or momentum, as presents in riverbed-wave propagation. Meanwhile, diffusion it self consists of diffusion of mass, diffusion of momentum and diffusion of heat (energy). In case of riverbed propagation the phenomenon can be approached by the phenomena of diffusion of mass.

As an explanation of the diffusion process, Graf (1998) presented transport process on molecular diffusion. Figure 1 shows the transport by convection-diffusion process. A point source of an extensive property, c_f (x,y;t), grows in forming a cloud, increasing in size and is responsible for energy dissipation within the fluid.

In case of riverbed wave, diffusion process is driven by friction force (shear stresses) and energy dissipations as represented in right-hand term of the equation (9). Since the energy dissipation is represented by behaviour of the velocity, then Vreugdenhil (1982) stated that diffusion coefficient could be defined as in equation (14).

Meanwhile, convection process is driven by celerity of the fluid motion. The celerity of the riverbed wave is represented by c_3 which shown in equation
(12a):

$$c_3 = \left[\frac{gdf(u)/du}{(gq/u^2) - u} \right] \quad \text{(12a)}$$

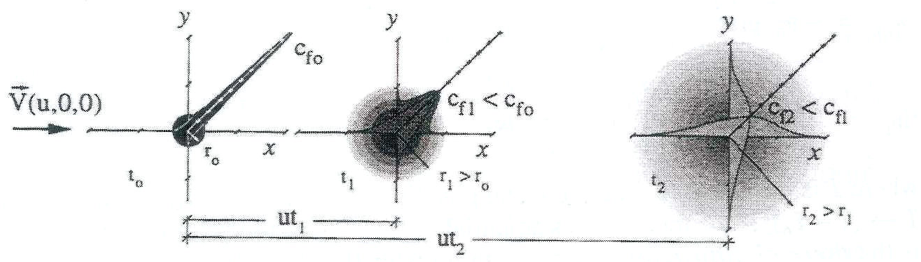

Figure 1. Convection-diffusion process (after Graf, 1998)

Since the riverbed celerity and diffusion factor has been defined, the convection-diffusion equation can be formed from equation (12). Using value of D deriving from the equation (14), which used as an approximation of the diffusion factor, the equation (12) can be transfer into:

$$\frac{\partial z}{\partial t} + c_3 \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(D \frac{\partial z}{\partial x} \right) \quad \text{(16a)}$$

Meanwhile, if the diffusion factor (D) is considered as a constant value, then the equation [16a] can be written as:

$$\frac{\partial z}{\partial t} + c_3 \frac{\partial z}{\partial x} = D \frac{\partial^2 z}{\partial x^2} \quad \text{(17)}$$

The equation (17) is known as the one-dimensional equation for convection-diffusion phenomena.

The interesting question is how do we know that such wave travels as a pure diffusion, as a pure convection or even as a convection-diffusion. The answer of the question above is very important as a guidance of applying proper analytical equation. To answer the question, Vreugdenhil (1982) stated that the Péclet number P = cL/D is the relevant parameter that can be used to check whether the system behaves as a pure diffusion equation, a pure convection or as a fully dynamic wave equation (convection-diffusion). On his paper he explained that for small value of *k*D/c the system behaves as a pure diffusion, on the other hand for large value of *k*D/c, the system behaves as a pure convection or mostly called a simple wave.

Riverbed Propagation

In this research, Instantaneous process was considered in order to study basic phenomena regarding riverbed-wave propagation system, particularly to study process of convection-diffusion within the system.

For pure diffusion case, Rath(2001) studied that for data related to the River Waal, Péclet number is 0.28 where L = 3500 m, D = 1.37 m2/s, c = 1.9E-04 m/s,

and k approximated by i/L is equal to 0.00012/3500.= 3.428E-08. If this assumption is applied then the pure diffusion equation will be enough to explain the system.

The pure diffusion equation based on equation (17) can be written as the following:

$$\frac{\partial z}{\partial t} = D\frac{\partial^2 z}{\partial x^2} \quad \text{....................................(18)}$$

In order to get general solution for the equation [18] in term of bed level (z), Wahono (2002) develop general solution based on solution presented by Graf (1998) for similar case. However, new boundary conditions were performed as the following:

$z(\pm\infty, t) = 0$
$z(x,0) = z_o$,

Using the boundaries above, and then transforming the equation [18] using Fourier transformation, the solution in term of z will be:

$$z(x,t) = \frac{z_o}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right) \quad \text{..... (19)}$$

In which the z_o is the integral of the wave area (m^2):

$$z_o = \int_{-\infty}^{+\infty} z(x)dx \quad \text{........................... (19a)}$$

However, in all cases where a local change is introduced into a river (such as a dam or local water withdrawal, which is also occur due to constriction), '*wave-length*' of the riverbed will be very small initially so that kD/c is large and the simple-wave character is then dominant.(Vreugdenhil,1982). In addition to that, in reality the riverbed wave behaves as non-linear phenomena.

Graf (1998) stated that in case of pure diffusion, the value diffuses in a stagnant medium, $\tilde{c}_3$=0 or motionless fluid. In case of pure convection, the value displaces itself with a velocity of translation $\tilde{c}_3 \neq 0$. When the $\tilde{c}_3 \neq 0$ and the diffusion factor is not equal to zero, then the fully convection-diffusion equation has to be used to explain the behaviour of the wave. Therefore, the equation [17] more properly applied for the convection-diffusion since celerity of riverbed also taken into account as convection parameter.

Using similar approach as pure diffusion equation, solution for the convection-diffusion was also developed by setting

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the new boundaries and deriving the general solution using the Fourier transformation. The boundaries was formed as the following:

z($\pm\infty$,t) = 0
z(x,0) = zo,

Using the boundaries above, using Fourier transformation, this study proposed the solution in term of z as the following:

$$z(x,t) = \frac{z_o}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x - c_3 t)^2}{4Dt} \right) \text{......(20)}$$

The equation (20) indicates that the value of z displace itself with the velocity of translation, $\vec{c}_3$, and at the same time it spreads out according to the normal curve. The peak z is propagated with velocity and it decreases with time. This phenomenon can be seen graphically in figure 2.

The equation of (19) and (20) are general solution for the instantaneous of occurrence. For the case of riverbed's discontinuities, it happened due to varying discharge and particularly during the high discharge, which is usually occurs for the certain limited time. Therefore, modification has to be done to get the proper solution.

The following calculations are regarding to the case of the River Waal in the Netherlands. The data assumptions are (Rath, 2001):

D (diffusion factor) = 1.373 m2/s
C_3 riverbed celerity = 1.095E-04 m/s
P (Péclet number) = 0.28

Using equation [20] the graphical view of the calculation for instantaneous of occurrence is shown in figure 3, which is corresponds to the diffusion effect. As an additional overview, Wahono (2002) presented the river bed-wave phenomena using numerical approach in different but similar case. The result shows that equation of convection-diffusion was properly used to explain the wave propagation within the main river channel. The graphs of those propagations are presented in figure 4.

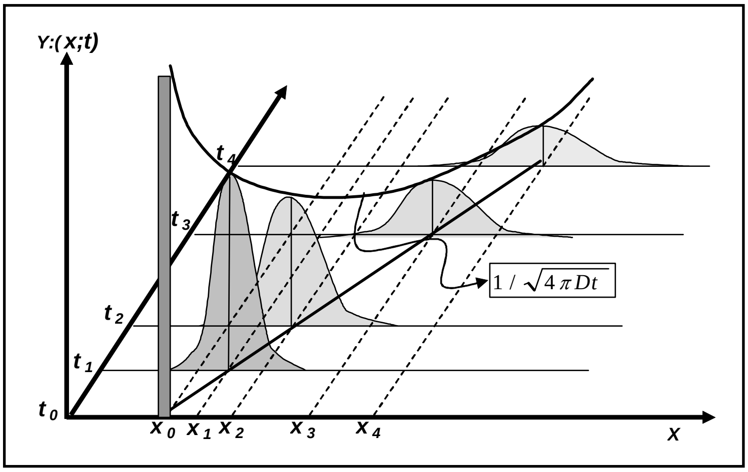

Figure 2. Convection-diffusion wave

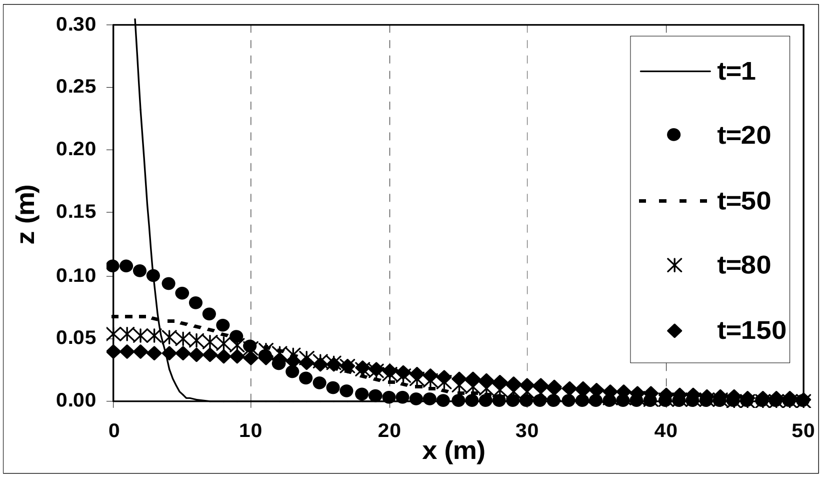

Figure 3. Distance related view of the wave

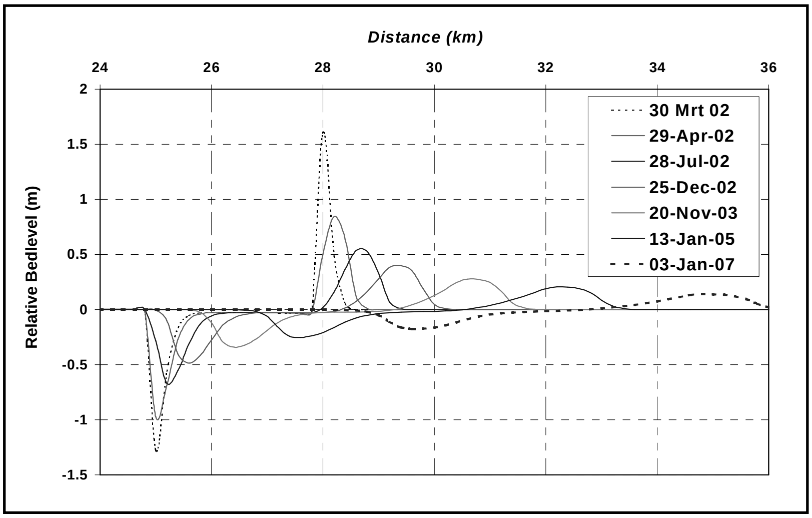

Figure 4. Riverbed propagation using numerical approach (after Wahono, 2002)

CONCLUSIONS

After studying the result, following conclusions can be derived:
Previous study presented that the behaviour of river bed-wave could be considered as a convection-diffusion process.

New general solution for convection-diffusion process for river bed-wave was proposed, based on original solution written by Graf (1998). Four basic equations, which consist of two equations of water movement and two equations of sediment movement, were used to derive the new general solution of the riverbed wave. There are two most importance variables in governing the convection-diffusion equation: celerity (c_3) and the diffusion factor (D). However, during the mathematical derivation, both c_3 and D were assumed as constant values. In early stages of the wave, process diffusion was dominant, whereas in later stages convection becomes more important.

It seems that analytical methods can be used for the derivation of such a method, though properly checking with a numerical model or any other method will be needed. The awareness of the applicability and limitation of the use of analytical approaches for this type of studies has to be addressed.

REFFERENCES

1. FISCHER, H. B., (1968), *Dispersion Predictions in Natural Stream*, Journal of Sanitary engineering Div., American Society of Civil Engineer, No.94(SA5): pp 927-944
2. GRAF, W.H and M.S. Altinakar, (1998), *Flow and Transport Processes in Channel of Simple Geometry*, John Willey & Sons, New York, 568 pages
3. RATH, S., (2001), *River Morphological Changes Induced by Spatial Variations in Floodplain Dimensions*, TUHH Germany, 51 pages
4. THOMANN, R.V., (1973), *Effect of Longitudinal Dispersion on Dynamic Water Quality Response of Stream and Rivers*, Water res. Research Journal, No. 9(2):pp 355-366.
5. VAN DER KLIS, H., (2000), *Stochastic modelling of river morphology: a case study*, Proceedings of the 8th international symposium on stochastic hydraulics, Beijing
6. VREUGDENHIL, C.B., (1982), *Numerical effects in models for river morphology*, In: Abbott, M.B. and J.A. Cunge, (1992), Engineering Applications of Computational hydraulics Vol. 1, Boston, Pitmant Publ. CY., pp 91 – 110
7. VRIES, M de, (1985), *A sensitivity analysis applied to morphological computations*, Communications on Hydraulics Dept of Civil Eng. Delft Univ. of Technology Report no. 85-2,31 pages.
8. WAHONO, E. P., (2002), *Behaviour of Riverbed Disturbances due to Floodplain Re-Naturalisation and Discharge Variation*, Report No.412 UNESCO-IHE Water Institute, Delft-The Netherlands. 212 pages.