JURNAL MATEMATIKA DAN KOMPUTER
Vol. 7. No. 2, 36 - 41, Agustus 2004, ISSN : 1410-8518

INTERPOLATION THEOREM FOR NONCOMMUTATIVE STANDARD EXTENSIONS OF LOGIC BB'I

Bayu Surarso
Department of Mathematics
Faculty of Mathematics and Natural Sciences
Diponegoro University

Abstract

Maehara in [Maehara, 1960/1961] introduces a proof-theoretical method for proving the interpolation theorem for standard logics. In the present paper we modify Maehara's method to prove the interpolation theorem for the systems **LBB'IK**, **LBB'IW** and **LBB'IKW** introduced in [Bayu Suraraso, 2005] and consequently the interpolation theorem holds for the logics **BB'IK**, **BB'IW** and **BB'IKW**.

Keywords: interpolation theorem, Maehara's method, **BB'IK**, **BB'IW** and **BB'IKW**.

1. INTRODUCTION

Let the expression V(D) denote the set of propositional variables which occur in the formula D. Interpolation theorem state the folowing property: *Suppose the formula $A \supset B$ is provable. Then there exists a formula C such that $V(C) \subset [V(A) \cap V(B)]$, for which both $A \supset C$ and $C \supset B$ are provable.*

Using cut eliminaton theorem, Maehara in [Maehara, 1960/1961] introduces a proof-theoretical method for proving the interpolation theorem for standard logics. Now it's well known as Maehara's method. (Detail proof of the interpolation method for standard logics, using Maehara's method, can be seen for example in [Takeuti, 1975].) By this method or just a minor modifications of it, **FL** and some of its contractionless extensions such as $\mathbf{Fl_e}$, $\mathbf{FL_w}$ and $\mathbf{Fl_{e,w}}$ can be shown to enjoy the interpolation theorem (See for example [Ono and Komori, 1985]).

In [Komori, 1994], Komori introduces a Gentzen-type formulation LBB'I for BB'I and proves its cut elimination theorem. Using a slight modification of Maehara's method he then shows that the interpolation theorem holds for **BB'I.** In [Bayu Surarso, 2005] the author introduces Gentzen-type formulations for **BB'IK, BB'IW** and **BB'IKW**. In the present paper, we will extend and modify Komori's proof of interpolation theorem for **BB'I** to show the interpolation theorem for **BB'IK, BB'IW** and **BB'IKW**.

We will follow notations used in [Bayu Surarso, 2005]. Thus we assume here a familiarity with Gentzen-type formulations **LBB'IK, LBB'IW** and **LBB'IKW** for **BB'IK, BB'IW** and **BB'IKW**, respectively.

2. MAEHARA'S METHOD

Before showing the interpolation theorem for noncommutative standard extensions of logic **BB'I**, let us first consider the idea of the original Maehara's method for proving the interpolation theorem for intuitionistic propositional logic. In the following, the Greek capital letters T, Δ, Σ, Π_1, … will denote finite sequences of formulas separated by commas. Let the expression V(T) denote the set of propositional variables which occur in the sequence T. We define partitions of sequence Γ as follows. Suppose Γ_1 is a sequence of some formula-occurences in Γ and suppose Γ_2 is the sequences of formula-occurences in Γ except those in Γ_1. Then we call $([\Gamma_1];[\Gamma_2])$ a partition of Γ. For example, ([A,A,C];[D,B]) is a partition of A,B,A,C,D. Next we prove that the following statement holds for the intuitionistic propositional logic.

Let $\Gamma\to D$ be a provable sequent and $([\Gamma_1];[\Gamma_2])$ be an arbitrary partition of Γ. Then there exist a formula C, called an interpolant of $\Gamma\to D$, such that

1) $\Gamma_1\to C$ and $C,\Gamma_2\to D$ are both provable,

2) $V(C)\subset V(\Gamma_1)\cap[V(\Gamma_2)\cup V(D)]$.

This statement is proved by induction on the number of inferences *k* in a *cut-free* proof of $\Gamma\to D$. For example, for the case $k > 0$ and the last inference is $(\to\wedge)$ as follow

$$\frac{\Gamma\to A \quad \Gamma\to B}{\Gamma\to A\wedge B}(\to\wedge).$$

By the hypothesis of induction, there are formulas C_1 and C_2 such that

1) $\Gamma_1\to C_1$ and $C_1,\Gamma_2\to A$ are both provable,

2) $V(C_1)\subset V(\Gamma_1)\cap[V(\Gamma_2)\cup V(A)]$

and

JURNAL MATEMATIKA DAN KOMPUTER
Vol. 7. No. 2, 36 - 41, Agustus 2004, ISSN : 1410-8518

3) $\Gamma_1 \rightarrow C_2$ and $C_2, \Gamma_2 \rightarrow B$ are both provable,

4) $V(C_2) \subset V(\Gamma_1) \cap [V(\Gamma_2) \cup V(B)]$.

From 1) and 3), using $(\rightarrow \wedge)$, $(\wedge \rightarrow 1)$ and $(\wedge \rightarrow 2)$, the sequent $\Gamma_1 \rightarrow C_1 \wedge C_2$ and $C_1 \wedge C, \Gamma_2 \rightarrow A \wedge B$ can be derived. From 2) and 4), it can be easily seen that $V(C_1 \wedge C_2) \subset V(\Gamma_1) \cap [V(\Gamma_2) \cup V(A \wedge B)]$. Thus $C_1 \wedge C_2$ become an interpolant of $\Gamma \rightarrow D$.

From the above statement, by taking a single formula for Γ and empty sequence for Γ_2, then it follows that the interpolation theorem holds for intuitionistic propositional logic.

3. INTERPOLATION THEOREM FOR BB'IK, BB'IW AND BB'IKW

In the present section we will show the interpolation theorem for **BB'IK, BB'IW** and **BB'IKW.** By the lack of exchange rule, the original Maehara's method will not work well for both of them. Then we will modificate it. By this modification we will see that the existence of guard-merge "$\circ$" in their Gentzen-type formulations gives no difficulties.

We prove first the interpolation theorem for **BB'IK** by the help of Gentzen-type system **LBB'IK.** To prove the theorem we need the following lemma.

Lemma 1. *Suppose that the sequent* $\Delta \circ \Gamma, \Sigma \rightarrow D$ *is provable (in* **LBB'IK)**, *where* Γ *is not void. Then there exists a formula C such that :*

1) $\Gamma \rightarrow C$ is provable,

2) $\Delta, C, \Sigma \rightarrow D$ is provable,

3) $V(C) \subset V(\Gamma) \cap V(\Delta, \Sigma, D)$.

Proof. We prove the lemma using induction on the number of inferences *k,* in a cut-free proof $\Delta \circ \Gamma, \Sigma \rightarrow D$. Here we will show only for the case when the last inference of the proof **P** of $\Delta \circ \Gamma, \Sigma \rightarrow D$ is weakening rule since the remaining cases can be treated essentially in the same way as the proof for the interpolation theorem for **BB'IK** in [Komori, 1994].

Suppose that the principal formula A of this application of weakening rule appears in Γ, that is $\Gamma = \Gamma_1, A, \Gamma_2$. Then the last part of **P** will be of the form $\dfrac{\Delta \circ (\Gamma_1, \Gamma_2), \Sigma \to D}{\Delta \circ (\Gamma_1, A, \Gamma_2), \Sigma \to D}(\text{weak})$.

Let us consider the upper sequent. By the hypothesis of induction there exists a formula C' such that:

1) $\Gamma_1, \Gamma_2 \to C'$ is provable,

2) $\Delta, C', \Sigma \to D$ is provable,

3) $V(C') \subset V(\Gamma_1, \Gamma_2) \cap V(\Delta, \Sigma, D)$.

From 1), by an application of wekening rule, we can get the following proof of $\Gamma \to C'$:

$$\dfrac{\dfrac{\vdots}{\Gamma_1, \Gamma_2 \to C'}}{\Gamma_1, A, \Gamma_2 \to C'}(\text{weak})$$

Next, obviously $V(\Gamma_1, \Gamma_2) \subset V(\Gamma_1, A, \Gamma_2)$. Then from 3) we can easily seen that $V(C') \subset V(\Gamma_1, \Gamma_2) \cap V(\Delta, \Sigma, D)$. Here C' become the interpolation of $\Delta \circ \Gamma, \Sigma \to D$.

In the same way we can get the interpolation $\Delta \circ \Gamma, \Sigma \to D$ for the case the last inference of the proof **P** of it is weakening rule and the principal formula A appears in Δ or in Σ.

Then by the above lemma we get,

Theorem 2. *The interpolation theorem holds **BB'IK**. More precisely, suppose A and B are formulas in **LBB'IK** such that $A \to B$ is provable. Then there exists a formula C such that both $A \to C$ and $C \to B$ are provable (in **LBB'IK**) and* $V(C) \subset V(A) \cap V(B)$.

Proof. Note that by using cut rule we can easily show that the formula $A \supset B$ is provable if and only if the sequent $A \to B$ is provable. Now, take Δ is empty, Γ = A, Σ is empty and D = B. Then by Lemma 1, if $A \to B$ is provable then there exists a formula C such that

1) $A \to C$ is provable,

2) $C \to B$ is provable,

3) $V(C) \subset V(A) \cap V(B)$.

Similarly to the proof of the interpolation theorem for **BB'IK**, we can also show that the interpolation theorem holds for **BB'IW**. In fact, Lemma 1 for **BB'IW** can be proved similarly to that of **BB'IK.** Note however that for the case when the last inference is contraction rule, the last part of proof P of $\Delta \circ \Gamma, \Sigma \to D$ may be as the following form $\dfrac{(\Delta_1, A, \Delta_2) \circ (\Gamma_1, A, \Gamma_2), \Sigma \to D}{(\Delta_1, \Delta_2) \circ (\Gamma_1, A, \Gamma_2), \Sigma \to D}(\text{con})$,

Here $\Gamma = \Gamma_1, A, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$. However, since in the application of (con) the indicated occurrences of the formula A in the antecedent of the upper sequent must occur consecutively, then we can also write this by the following form

$$\frac{\mu(\Delta_1, \Gamma_1), A, A, \Delta_2 \circ \Gamma_2, \Sigma \to D}{\mu(\Delta_1, \Gamma_1), A, \Delta_2 \circ \Gamma_2, \Sigma \to D}(\text{con})$$

where $\mu(\Delta_1, \Gamma_1)$ denotes the sequences obtained by merging Δ_1 and Γ_1. Then we can find the interpolant of $(\Delta_1, \Delta_2) \circ (\Gamma_1, A, \Gamma_2), \Sigma \to D$ by the following way.

First let us consider the upper sequent. By the hypothesis induction there exists a formula C' such that

1) $\mu(\Delta_1, \Gamma_1), A, A, \Delta_2 \circ \Gamma_2, \Sigma \to C'$ is provable,

2) $C', \Sigma \to D$ is provable,

3) $V(C) \subset V(\mu(\Delta_1, \Gamma_1), A, A, \Delta_2 \circ \Gamma_2) \cap V(\Sigma, D)$.

From 1), by an application of contraction rule, we can get the following proof of $\mu(\Delta_1, \Gamma_1), A, \Delta_2 \circ \Gamma_2 \to C'$

$$\cfrac{\cfrac{\vdots}{\mu(\Delta_1, \Gamma_1), A, A, \Delta_2 \circ \Gamma_2, \Sigma \to C'}}{\mu(\Delta_1, \Gamma_1), A, \Delta_2 \circ \Gamma_2, \Sigma \to C'}(\text{con})$$

Next, it is obvious that $V(\mu(\Delta_1, \Gamma_1), A, A, \Delta_2 \circ \Gamma_2) = V(\mu(\Delta_1, \Gamma_1), A, \Delta_2 \circ \Gamma_2)$. Then from 3) we can easily seen that $V(C') \subset V(\mu(\Delta_1, \Gamma_1), A, \Delta_2 \circ \Gamma_2) \cap V(\Sigma, D)$. Here C' become the interpolant of $\mu(\Delta_1, \Gamma_1), A, \Delta_2 \circ \Gamma_2, \Sigma \to D$.

By the similar way, we can also find the interpolant of $\Delta \circ \Gamma, \Sigma \to D$ for the case when the last part of the proof **P** is as the forms

$$\frac{(\Delta_1, A, \Delta_2) \circ (\Gamma_1, A, \Gamma_2), \Sigma \to D}{(\Delta_1, A, \Delta_2) \circ (\Gamma_1, \Gamma_2), \Sigma \to D}(\text{con}), \qquad \frac{\Delta \circ (\Gamma', A), A, \Sigma \to D}{\Delta \circ (\Gamma', A), \Sigma \to D}(\text{con}) \qquad \text{and}$$

$$\frac{\Delta \circ (\Gamma, A), (A, \Sigma') \to D}{\Delta \circ \Gamma, (A, \Sigma') \to D}(\text{con})$$

Now, since Lemma 1 holds for **LBB'IW**, then by the same way to the proof of Theorem 2 we get the following

Theorem 3. *The interpolation theorem holds **BB'IW**. More precisely, suppose A and B are formulas in **LBB'IW** such that A $\to$ B is provable. Then there exists a formula C such that both A $\to$ C and C $\to$ B are provable (in **LBB'IW**) and* $V(C) \subset V(A) \cap V(B)$.

Finally, from the proofs of Theorem 2 and Theorem 3 we directly get the following theorem

Theorem 4. *The interpolation theorem holds **BB'IKW**. More precisely, suppose A and B are formulas in **LBB'IKW** such that A $\to$ B is provable. Then there exists a formula C such that both A $\to$ C and C $\to$ B are provable (in **LBB'IKW**) and* $V(C) \subset V(A) \cap V(B)$.

REFERENCES

Bayu Surarso, *Gentzen-type systems for logic **BB'I** and its noncommutative standard extensions*, Journal of Sciences & Mathematics (JSM), Fakultas MIPA UNIVERSITAS DIPONEGORO, Semarang, 2004, Vol. 12, No. 3.

Komori, Y., *Syntactical investigations in **BI** logic and **BB'I** logic*, Studia Logica, 1994, Vol. 53, pp.397-416.

Maehara, S., *Craig no interpolation theorem* (in Japanese), Suugaku, 1960/1961, Vol. 12, pp. 235-237.

Ono, H. and Komori, Y., *Logics without the contraction rule*, Journal of Symbolic Logic, 1985, Vol. 50, pp.169-201.

Takeuti, G., Proof Theory, North-Holland, Amsterdam, 1975.