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ANALYSIS OF MULTI-OBJECTIVE LINEAR ROBUST OPTIMIZATION MODEL WITH LEXICOGRAPHICAL METHOD

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Abstract: Problems in robust multi-objective linear optimization are a class of optimization problems with uncertain data parameters which aim in the decision-making process to obtain the best results in certain circumstances by choosing various solution methods for the multi-objective. This research aims to formulate a multi-objective Robust Optimization (RO) model using the Lexicographic Method, then analyzing the existence and uniqueness of the solution. Furthermore, gap analysis on the topic was carried out using a Systematic Literature Review (SLR) approach with the Preferred Reporting Items for Systematic Review and Meta Analysis (PRISMA) method. Results in SLR, the analysis results also shows that the Lexicographic Method is effective in handling data uncertainty with the objective functions sorted by priority. The robust formulation with polyhedral uncertainty sets ensures the flexibility and adaptability of the model. Convexity analysis and application of the Karush-Kuhn-Tucker (KKT) method prove that the resulting solution is exist and unique.

1. INTRODUCTION

Optimization is the act of decision-making to get the best results in certain circumstances (Talatahari & Azizi, 2020). In mathematics, optimization is a process of achieving the best conditions that provide a minimum or maximum value of a function that is limited by certain circumstances (Rao, 2009). In real life, this problems are often faced with the problem of more than one objective function, which is called multi-objective optimization (Sharma & Chahar, 2022). Multi-objective optimization is a field in optimization that aims to find optimal solutions that involve multi-objective functions (Shi et al., 2023), and under a series of constraints (Rao, 2009). Solving multi-objective optimization problems requires a method, one of which is Lexicographic Method.

Lexicographic Method is an approach to multi-objective optimization in which are given a priority order (Rao, 2009). By ordering these objectives, the resulting solution can reflect the preferences of decision makers who consider the order of priorities (Gheouany et al., 2023). In real life, optimization problems often experience problems with data that cannot be known with certainty, which is called data that contains uncertainty (Chaerani et al., 2022). One of the areas of optimization that can be applied to solve problems related to uncertainty is RO (Chaerani et al., 2021).

RO is a methodology for dealing with problems affected by data uncertainty where there is no distribution of opportunities that satisfy the uncertainty of the parameters (Gabrel et al., 2014). To overcome this data uncertainty, the methodology uses Robust Counterpart (RC), which is assumed to be in a series of uncertainties, one of which is polyhedral (Ben-Tal & Nemirovski, 2002). In recent years, developments regarding uncertainty in multi-objective optimization problems have been carried out by several researchers, including Chaerani et al. (2021) who discuss the development of a robust multi-objective optimization model to solve spatial land use allocation problems using lexicographic methods. Apart from that, research conducted by Muslihin et al. (2022) discusses Conic Duality research regarding multi-objective strong optimization problems. In contrast to previous studies, this research includes the formulation of RC model for multi-objective linear optimization model problems using the Lexicographic Method where the existence and uniqueness of solutions are analyzed using convex analysis and KKT method. However, a gap analysis of the problem is first carried out using SLR with PRISMA method.

Convex analysis is a branch of mathematical analysis that deals with set and function convectors in the analytical review of sets and functions (Singh & Singh, 2023). The concepts in convex analysis have many roles in various fields such as in the field of operations management (Chen & Li, 2021), matching mechanisms (Kojima et al., 2018) and optimization (Li & Mastroeni, 2020). While, the KKT method is a method with a series of conditions that is used to determine the optimal solution to minimize and maximize problems with constraints (Agarwal et al., 2023). The KKT method states that the optimal solution must meet several conditions, including complementary conditions (Tulshyan et al., 2010). With KKT, points where the constraints in the optimization problem have variable values in accordance, or not fulfil can be identified.

Based on the above discussion, the objectives of this study are: (1) to analyze the gap of multi-objective linear RO problems with the Lexicographic method which discusses the general model and its mathematical analysis, (2) to formulate a multi-objective linear RO model with the Lexicographic method, (3) to analyze the existence and solutions of the model formulation obtained.

2. LITERATURE REVIEW

2.1. Multi-objective Linear Optimization and Lexicographic Methods

Multi-objective optimization is the solution of optimization problems that have more than one objective function. For linear problem, the form of multi-objective optimization (Goberna et al., 2014):

$$\min.\{\boldsymbol{c}_1^T\boldsymbol{x}, \boldsymbol{c}_2^T\boldsymbol{x}, \dots, \boldsymbol{c}_k^T\boldsymbol{x}\} \qquad \text{s.t. } \boldsymbol{a}_j^T\boldsymbol{x} \geq b_j; j \in T \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n, T, \boldsymbol{c}_k \in \mathbb{R}^n; k = 1,2, \dots, i, a_j \in \mathbb{R}^n, b_j \in \mathbb{R}, \forall j \in T$.

Referring to Rao (2009), Lexicographic Method is a method that orders objective functions based on their priority determined by the researcher. Suppose $f_1(x)$ and $f_k(x)$ represent the most important objective function and the least important solution and $\boldsymbol{x}^*$ is the optimal solution obtained. Then, the problem i-th is as follows:

$$\min f_l(\boldsymbol{x}) \qquad \text{s.t.: } g_p(x) \leq 0, p = 1, 2, \dots, m,$$
$$f_l(x) = f_l^*, l = 1, 2, \dots, (i-1) \tag{2}$$

The solution is obtained, namely $\boldsymbol{x}_i^*$ and f_l^*=$f_l(\boldsymbol{x}_l^*)$. The final solution to the problem $\boldsymbol{x}_\mathrm{k}^*$, got by the solution $\boldsymbol{x}^*$from the initial problem of multi-objective optimization.

2.2. Robust Optimization (RO)

According to Ben-Tal & Nemirovski (2002) RO is an optimization field that tackles optimization problems involving uncertainty parameters. A general form for linear optimization problems with uncertainty in vectors $\boldsymbol{c}$, $\boldsymbol{b}$ and $\boldsymbol{A}$ (Ben-Tal et al., 2009; Gorissen et al., 2015):

$$\min_{x}\{\boldsymbol{c}^T\boldsymbol{x}: \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} | (\boldsymbol{c}, \boldsymbol{A}, \boldsymbol{b}) \in \mathcal{U}\} \tag{3}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\mathcal{U}$ is uncertainty set.

RO is based on three basic assumptions in decision-making (Gorissen et al., 2015) namely all decision variables represent decisions "here and now", decision-makers are responsible for decisions taken with the state of parameters being in the set of uncertainties and constraints in the problem RO is a "hard $\mathcal{U}$, constraint". In addition to these basic assumptions, the general model of RO problems has additional basic assumptions (Gorissen et al., 2015):

A.1 If there is uncertainty in the objective function $\boldsymbol{c}^T\boldsymbol{x}$, then $\boldsymbol{c}^T\boldsymbol{x}$ in (3) can be replaced by additional variables $t \in \mathbb{R}$ where $t \geq \boldsymbol{c}^T\boldsymbol{x}$, and t in (4) moved to the left segment so that it is obtained:

$$\min_{\boldsymbol{x},t}\{t: \boldsymbol{c}^T\boldsymbol{x} - t \leq 0, \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} | (\boldsymbol{c}, \boldsymbol{A}, \boldsymbol{b}) \in \mathcal{U}\} \tag{4}$$

A.2 If there is uncertainty in parameter $\boldsymbol{b}$, then the parameters $\boldsymbol{b}$ in (3) can be multiplied by variables x_{n+1} where $x_{n+1} = 1$, variables $\boldsymbol{x}$ and t moved to the left side so that it is obtained:

$$\min_{\boldsymbol{x},t}\{\boldsymbol{c}^T\boldsymbol{x}: \boldsymbol{a}_i^T\boldsymbol{x} - b_i x_{n+1} \leq 0, x_{n+1} = 1, i = 1,2, \dots, m, (\boldsymbol{A}, \boldsymbol{b}) \in \mathcal{U}\}. \tag{5}$$

A.3 If $\boldsymbol{x}$ is a solution for Robust, then $\boldsymbol{x}$ is still a viable solution when the set of uncertainties $\mathcal{U}$ changed to *convex hull* $\mathcal{U}$. *Convex hull* $\mathcal{U}$ is the smallest convex set of $\mathcal{U}$.

A.4 Robustness towards $\mathcal{U}$ can be formulated in *constraint-wise*, that is Robust*ness* the problem of RO can be seen in each constraint with a set of uncertainties $\mathcal{U}$ closed and convex.

The RC reformulations for polyhedral is:

$$\boldsymbol{a}^T\boldsymbol{x} + \boldsymbol{q}^T\boldsymbol{y} \leq \boldsymbol{b} \qquad D^T\boldsymbol{y} = -P^T\boldsymbol{x} \qquad \boldsymbol{y} \geq 0 \tag{6}$$

2.3. Convex Analysis

Definition 1 (Bartle and Sherbert, 2010). *Let $I \subseteq \mathbb{R}$ is an interval, let $f: I \longrightarrow \mathbb{R}$, and let $c \in I$. Real number L is derivative of f at point c, if for any $\varepsilon > 0$, there exists $\delta > 0$ such as:*

$$\forall \boldsymbol{x} \in I, 0 < |x - c| < \delta \Longrightarrow \left|\frac{f(x) - f(c)}{x - c} - L\right| < \varepsilon \tag{7}$$

Thorem 1 (Bartle and Sherbert, 2010). *If $f: I \to \mathbb{R}$ has a derivative at $c \in I$, then f continuous at c.*

Definition 2 (Brinkhuis, 2020). *Let $\boldsymbol{A} \subseteq \mathbb{R}^n$ is nonempty convex set. Function $f: \boldsymbol{A} \to \mathbb{R}$, it says convex function if for every $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{A}$ and $\lambda \in [0, 1]$ then:*

$$f(\lambda\boldsymbol{x} + (1 - \lambda)\,\boldsymbol{y}) \leq \lambda\, f(\boldsymbol{x}) + (1 - \lambda) f(\boldsymbol{y}) \tag{8}$$

Definition 3 (Boyd and Vandenberghe, 2004). *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ It is said to be affine if it is the sum of a linear function and a constant, i.e. f has a form:*

$$f(x) = \boldsymbol{A}x + b \qquad \text{with } \boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{x} \in \mathbb{R}^n, \text{dan } \boldsymbol{b} \in \mathbb{R}^m. \tag{9}$$

2.4. Karush-Kuhn-Tucker (KKT) Method

KKT is a method for solving optimization problems in order to find the optimal solution of a function with constraints. For optimization problems with inequality and equality constraint functions, suppose functions $f(\boldsymbol{x}), g_i(\boldsymbol{x}), h_j(\boldsymbol{x})$ is a differentiable and continuous functions. Let $\hat{x}$ is a fissible point that solves the above problem locally with $\nabla g_i(\widehat{\boldsymbol{x}})$ and $\nabla h_j(\widehat{\boldsymbol{x}})$ is constraint function. Then there is a scalar λ_i and μ_j the so-called lagrange caller is such that:

$$\nabla f(\widehat{\boldsymbol{x}}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\widehat{\boldsymbol{x}}) + \sum_{j=1}^{p} \mu_j \nabla h_j(\widehat{\boldsymbol{x}}) = 0 \quad (10)$$

where the constraints are:

$$g_i(\widehat{\boldsymbol{x}}) \leq 0, \qquad i = 1, 2, \ldots, m \quad (11)$$
$$h_j(\widehat{\boldsymbol{x}}) = 0, \qquad j = 1, 2, \ldots, p \quad (12)$$
$$\lambda_i g_i(\widehat{\boldsymbol{x}}) = 0, \qquad i = 1, 2, \ldots, m \quad (13)$$
$$\lambda_i \geq 0, \qquad i = 1, 2, \ldots, m \quad (14)$$

It is called the complementary slackness condition which expresses two possibilities, namely:
1. If $g_i(\widehat{\boldsymbol{x}}) < 0$, then $\lambda_i = 0$,
2. If $\lambda_i > 0$, then constraints $g_i(\widehat{\boldsymbol{x}}) = 0$.

3. METHODOLOGY

3.1. Stages of Systematic Literature Review

SLR in this study used PRISMA (Moher et al., 2009), which provides guidance for conducting SLR (Stovold et al., 2014), and improves the quality of the methodology and results of the review (Panik et al., 2013). PRISMA has several stages including identification, screening, Eligibility, and included (Irmansyah et al., 2022; Utomo et al., 2018). The process on first step includes keyword classification is carried out using a combination of the keywords "Optimization Model", "Linear", " Robust Optimization ", "Lexicographic Method" and "Analysis". A literature search was carried out from four databases Scopus, Science Direct, Dimensions, and Google Scholar then checked for duplication using Mendeley software (Rathbone et al., 2015). The second step involved filtering articles by title and abstract, while the third step included a full content review, categorizing articles into two groups: 183 articles rated with one star and 19 with two stars. Dataset 1, consisting of these articles, was used for bibliometric mapping and thematic evolution analysis with RStudio. The final stage involved further analysis of the most relevant articles, referred to as dataset 2, to identify research gaps.

3.2. Stage of Multi-objective RO Model with Lexicographic Methods, Existence and Uniqueness Solution Analysis

The Problem Context begins with the input of an uncertainty optimization problem which is a special problem in a multi-objective robust linear optimization model solved using the Lexicographic Methods, which is that assumed the objective function coefficient and the constraint function which is assumed to be uncertainty set $\mathcal{U}$ on polyhedral uncertainty set. The solution of this stage is carried out by changing the uncertainty multi-objective linear optimization model to the RC optimization problem. After the model formulation is obtained, an analysis of the existence and solution is carried out. The model formulation obtained will be analyzed for the guarantee of existence and the uniqueness of the solution by the KKT Method and convex analysis.

4. RESULTS AND DISCUSSION

4.1 The Result of Systematic Literature Review

The Figure 1 shows results with four clusters based on color (green, red, blue, and purple), representing 50 articles. The keyword “RO” which is linked to “Multi-objective Optimization”, where this keyword is related to the keywords “Convex Optimization” and “Linear Programming”.

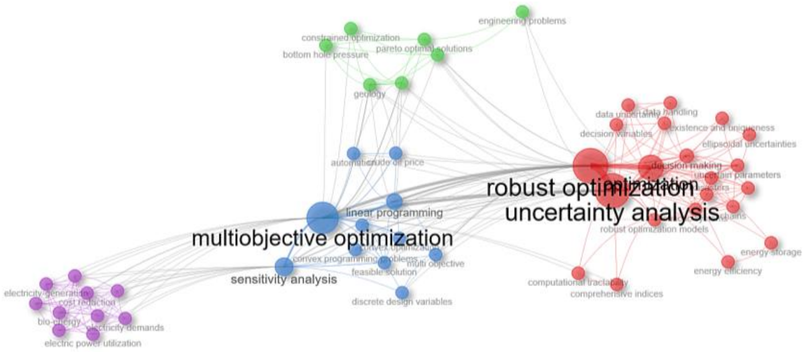

Figure 1. Bibliometric Map

The Figure 2 shows the frequency of the word “linear programming” with one event in 2014-2022, increased in 2014-2016 and 2017-2018. In 2021-2022, the frequency of the word “Multi-objective Optimization” was six. The frequency of the word “Optimization” appearing was two in 2020-2022 and increased in 2020-2021 with two frequencies. The frequency of the word “Robust Optimization” occurred in 2014-2023 with the highest frequency appearing in 2021-2022. However, from Figure 1 and 2 the keywords “Analysis”, “optimization model” and “Lexicographic Method” were not found, which could be a robust reason for this study to be conducted.

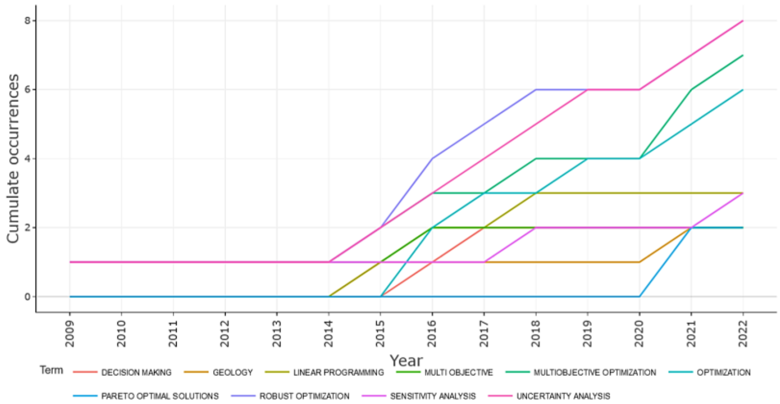

Figure 2. Word Frequency over Time

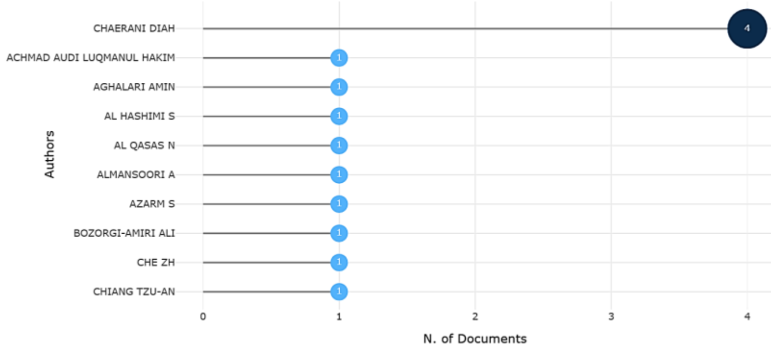

Figure 3. Most Relevant Authors

Evaluation of the relevance of article authors can be done by analyzing specifically about the contributions of each author. Figure 3 depicts 10 authors who have a significant influence on the topic that is currently being discussed systematically. In the Figure 3, there is one most influential author with four published articles, followed by nine other authors with one published article.

In Table 1, six articles relevant to the study are highlighted. Siraj et al. (2015) examined an economic optimization model based on oil production using the Lexicographic Method. Wang and Fang (2018) proposed a multi-objective linear programming model incorporating uncertainty with ellipsoidal and general norm uncertainty sets. Perdana et al. (2020) applied the Lexicographic Method to address multi-objective optimization, focusing on balancing demand fulfillment and logistics costs. Chaerani et al. (2021) explored a robust multi-objective optimization model for spatial land use allocation, employing Lexicographic and uncertain methods within ellipsoidal and polyhedral uncertainty sets. Kecskés and Odry (2021) tackled multi-purpose scenario problems using a modified Lexicographic Method with a minimum-maximum scheme to achieve optimal solutions. Lastly, Muslihin et al. (2022) investigated the cone duality of multi-objective robust optimization problems, addressing them with the utility function method.

Tabel 1. State-of-the-art

Writer	Type		Uncertainty Set	Lexicographic Method	Analysis Review	
	Linear	Nonlinear			Existence & uniqueness of solutions	Convexity
Siraj et al. (2015)	-	√	-	√	-	-
Wang & Fang (2018)	√	-	Ellipsoidal & General Norm	-	-	-
Perdana et al. (2020)	√	-	Box	√	-	-
Chaerani et al. (2021)	√	-	Ellipsoidal & Polyhedral	√	-	-
Kecskés & Odry (2021)	-	√	-	-	-	-
Muslihin et al. (2022)	√	-	Ellipsoidal & Polyhedral	-	√	√

4.2. Formulation of RC Models for Multi-objective Robust Linear Optimization Problems

In this study, it is assumed that uncertainty is in the objective function coefficient (c_k) and the coefficient of the constraint function (A). Based on assumption A.1, the objective function can be replaced by an additional variable $t_k \in \mathbb{R}$ with $t_k \geq c_k^T \boldsymbol{x}$, so that it is obtained:

$$\min . t_k \quad \text{s.t.:} \quad c_k^T \boldsymbol{x} \leq t_k; \quad \boldsymbol{A}\boldsymbol{x} \leq b; \quad \boldsymbol{c}_l^T \boldsymbol{x} = \boldsymbol{c}_l^*, l = 1,2,\dots,(k-1); \quad \boldsymbol{x} \geq 0; \quad (\boldsymbol{c}_k, \boldsymbol{A}) \in \mathcal{U}. \tag{15}$$

Moving variables t_k on the first constraint to the left side, and change it to the standard form of variable linear programming, namely t_k is changed to a nonnegative variable, i.e. by assuming t_k as $t_k = \delta_k^{(1)} - \delta_k^{(2)}$, $\delta_k^{(1)}, \delta_k^{(2)} \geq 0$, so that (15) becomes:

$$\min . \delta_k^{(1)} - \delta_k^{(2)} \quad \text{s.t.:} \quad \boldsymbol{c}_k^T \boldsymbol{x} - \delta_k^{(1)} + \delta_k^{(2)} \leq 0; \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}; \quad \boldsymbol{c}_l^T \boldsymbol{x} = \boldsymbol{c}_l^*, l = 1,2,\dots,(k-1); \quad \delta_k^{(1)}, \delta_k^{(2)} \geq 0; \quad \boldsymbol{x} \geq 0; \quad (\boldsymbol{c}_k, \boldsymbol{A}) \in \mathcal{U}. \tag{16}$$

Furthermore, $\boldsymbol{c}_k$ and $\boldsymbol{A}$ is expressed in a primitive uncertainty parameter $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \in Z$ with $\boldsymbol{\zeta}_1 = \left(\zeta_1^{(1)}, \zeta_1^{(2)}, \dots, \zeta_1^{(w)}\right) \in \mathbb{R}^w$ and $\boldsymbol{\zeta}_2 = \left(\zeta_2^{(1)}, \zeta_2^{(2)}, \dots, \zeta_2^{(i)}\right) \in \mathbb{R}^i$. Model (16) is a problem that only has uncertainty in the coefficient of the first and second constraint. RC (16) becomes:

$$(\bar{\boldsymbol{c}} + P\boldsymbol{\zeta}_1)^T\boldsymbol{x} - \delta_k^{(1)} + \delta_k^{(2)} \leq 0, \qquad (\bar{\boldsymbol{a}} + Q\boldsymbol{\zeta}_2)^T\boldsymbol{x} \leq \boldsymbol{b}, \tag{17}$$

with $(\bar{\boldsymbol{c}} + P\boldsymbol{\zeta}_1)^T$dan $(\bar{\boldsymbol{a}} + Q\boldsymbol{\zeta}_2)^T$ is an affine function. Furthermore, formulation (17) equivalent:

$$\bar{\boldsymbol{c}}^T\boldsymbol{x} + (P^T\boldsymbol{x})^T\boldsymbol{\zeta}_1 - \delta_k^{(1)} + \delta_k^{(2)} \leq 0, \qquad \bar{\boldsymbol{a}}^T\boldsymbol{x} + (Q^T\boldsymbol{x})^T\boldsymbol{\zeta}_2 \leq \boldsymbol{b}, \tag{18}$$

Then, the determination of this problem solutions involving data uncertainty is "the best worst-case" solution, therefore reformulation was carried out in (19) as follows:

$$\bar{\boldsymbol{c}}^T\boldsymbol{x} + \max_{\zeta_1}(P^T\boldsymbol{x})^T\boldsymbol{\zeta}_1 - \delta_k^{(1)} + \delta_k^{(2)} \leq 0, \quad \bar{\boldsymbol{a}}^T\boldsymbol{x} + \max_{\zeta_2}(Q^T\boldsymbol{x})^T\boldsymbol{\zeta}_2 \leq \boldsymbol{b}, \tag{19}$$

4.2 Formulation Model for Multi-objective Linear Counterpart RO Problem with Polyhedral Uncertainty Set

The formulation of this optimization model assumes that the parameter is uncertainty c_k and A is in the set of polyhedral uncertainties defined as follows:

$$Z_1 = \{\boldsymbol{\zeta}_1 : \boldsymbol{n} - \boldsymbol{N}\boldsymbol{\zeta}_1 \geq 0\}, \qquad Z_2 = \{\boldsymbol{\zeta}_2 : \boldsymbol{h} - \boldsymbol{H}\boldsymbol{\zeta}_2 \geq 0\}, \tag{20}$$

$\boldsymbol{n} \in \mathbb{R}^j, \boldsymbol{N} \in \mathbb{R}^{w\times j}, \boldsymbol{h} \in \mathbb{R}^r, \boldsymbol{H} \in \mathbb{R}^{i\times r}, \boldsymbol{\zeta}_1 \in \mathbb{R}^w$, and $\boldsymbol{\zeta}_2 \in \mathbb{R}^i$.

Reformulating the left side in both constraints in (19), so that it is equivalent to:

$$\bar{\boldsymbol{c}}^T\boldsymbol{x} + \max_{\zeta_1 : \boldsymbol{n} - \boldsymbol{N}\zeta_1 \geq 0}(P^T\boldsymbol{x})^T\boldsymbol{\zeta}_1 - \delta_k^{(1)} + \delta_k^{(2)} \leq 0, \tag{21}$$

$$\bar{\boldsymbol{a}}^T\boldsymbol{x} + \max_{\zeta_2 : \boldsymbol{h} - \boldsymbol{H}\zeta_2 \geq 0}(Q^T\boldsymbol{x})^T\boldsymbol{\zeta}_2 \leq \boldsymbol{b}, \tag{22}$$

The constraint (21) has uncertainty variables $\boldsymbol{\zeta}_1$ that unrestricted with n moved to the right side and both sides are multiplied by (-1), obtained:

$$\max(\boldsymbol{P}^T\boldsymbol{x})^T\boldsymbol{\zeta}_1 \qquad \text{s.t.:} \quad \boldsymbol{N}\boldsymbol{\zeta}_1 \leq \boldsymbol{n}; \quad \boldsymbol{\zeta}_1 \in \mathbb{R}^w. \tag{23}$$

Using the robust duality theorem, the dual form of the problem (23) is as follows:

$$\min\ \boldsymbol{n}^T\boldsymbol{\gamma} \qquad \text{s.t.:} \quad \boldsymbol{N}^T\boldsymbol{\gamma} = \boldsymbol{P}^T\boldsymbol{x}; \quad \boldsymbol{\gamma} \geq 0. \tag{24}$$

with $\boldsymbol{\gamma} \in \mathbb{R}^j$. Substitution (24) to (21) results in:

$$\bar{\boldsymbol{c}}^T\boldsymbol{x} + \min_{\gamma}\{\boldsymbol{n}^T\boldsymbol{\gamma} : \boldsymbol{N}^T\boldsymbol{\gamma} = \boldsymbol{P}^T\boldsymbol{x}, \boldsymbol{\gamma} \geq 0\} - \delta_k^{(1)} + \delta_k^{(2)} \leq 0, \tag{25}$$

Problem (25) is met for a physical solution γ in the physical set $\mathcal{F}$. So, the constraint (25) becomes:

$$\bar{\boldsymbol{c}}^T\boldsymbol{x} + \boldsymbol{n}^T\boldsymbol{\gamma} - \delta_k^{(1)} + \delta_k^{(2)} \leq 0, \boldsymbol{N}^T\boldsymbol{\gamma} = \boldsymbol{P}^T\boldsymbol{x}, \boldsymbol{\gamma} \geq 0. \tag{26}$$

In the same way, a RC formulation for constraints is obtained (22):

$$\bar{\boldsymbol{a}}^T\boldsymbol{x} + \boldsymbol{h}^T\boldsymbol{\lambda} \leq \boldsymbol{b}, \boldsymbol{H}^T\boldsymbol{\lambda} = \boldsymbol{Q}^T\boldsymbol{x}, \boldsymbol{\lambda} \geq 0. \tag{27}$$

Constraints (26) and (27) are linear constraints so they are guaranteed to be computationally tractable. The primal-dual relationship above can be written as follows:

$$\max_{\zeta_1 : \boldsymbol{n} - \boldsymbol{N}\zeta_1 \geq 0}(\boldsymbol{P}^T\boldsymbol{x})^T\boldsymbol{\zeta}_1 = \min_{\gamma}\{\boldsymbol{n}^T\boldsymbol{\gamma} : \boldsymbol{N}^T\boldsymbol{\gamma} = \boldsymbol{P}^T\boldsymbol{x}, \boldsymbol{\gamma} \geq 0 \tag{28}$$

$$\max_{\zeta_2 : \boldsymbol{h} - \boldsymbol{H}\zeta_2 \geq 0}(Q^T\boldsymbol{x})^T\boldsymbol{\zeta}_2 = \min_{\lambda}\{\boldsymbol{h}^T\boldsymbol{\lambda} : \boldsymbol{H}^T\boldsymbol{\lambda} = \boldsymbol{Q}^T\boldsymbol{x}, \boldsymbol{\lambda} \geq 0 \tag{29}$$

Both dual formulations are met for a feasible solution γ and λ contained in the following:

$$\mathcal{F} = \{\boldsymbol{\gamma} : \boldsymbol{N}^T\boldsymbol{\gamma} = \boldsymbol{P}^T\boldsymbol{x}, \boldsymbol{\gamma} \geq 0\} \to \exists\boldsymbol{\gamma} \geq 0 \ni \boldsymbol{N}^T\boldsymbol{\gamma} = \boldsymbol{P}^T\boldsymbol{x}. \tag{30}$$

$$\mathcal{F} = \{\boldsymbol{\lambda}: \boldsymbol{H}^T\boldsymbol{\lambda} = \boldsymbol{Q}^T\boldsymbol{x}, \boldsymbol{\lambda} \geq 0\} \rightarrow \exists\boldsymbol{\lambda} \geq 0 \ni \boldsymbol{H}^T\boldsymbol{\lambda} = \boldsymbol{Q}^T\boldsymbol{x}. \quad (31)$$

Thus, the formulation of a multi-objective robust linear optimization model using the Lexicographic Method and assumed to be in the polyhedral uncertainty set is as follows:

$$\min.\ \delta_k^{(1)} - \delta_k^{(2)} \quad (32)$$

$$\begin{aligned}
\text{s.t.:}\quad & s.t.\ \bar{\boldsymbol{c}}^T\boldsymbol{x} + \boldsymbol{n}^T\boldsymbol{\gamma} - \delta_k^{(1)} + \delta_k^{(2)} \leq 0; && \boldsymbol{N}^T\boldsymbol{\gamma} = \boldsymbol{P}^T\boldsymbol{x};\\
& \bar{\boldsymbol{a}}^T\boldsymbol{x} + \boldsymbol{h}^T\boldsymbol{\lambda} \leq \boldsymbol{b}; && \boldsymbol{H}^T\boldsymbol{\lambda} = \boldsymbol{Q}^T\boldsymbol{x};\\
& \boldsymbol{c}_l^T\boldsymbol{x} = \boldsymbol{c}_l^*, l = 1,2,\dots,(k-1); && \boldsymbol{x} = (x_1, x_2, \dots, x_p) \geq 0;\\
& \delta_k^{(1)} = \left(\delta_1^{(1)}, \delta_2^{(1)}, \dots, \delta_m^{(1)}\right) \geq 0; && \delta_k^{(2)} = \left(\delta_1^{(2)}, \delta_2^{(2)}, \dots, \delta_m^{(2)}\right) \geq 0;\\
& \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_j) \geq 0; && \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_r) \geq 0.
\end{aligned}$$

The objective and constraint function in (32) are linear functions, so it is clear that the functions are differential and continuous.

4.3 Convexity Analysis on Objective Functions

Known the objective functions $f\left(\delta_k^{(1)}, \delta_k^{(2)}\right) = \delta_k^{(1)} - \delta_k^{(2)}$ that will be proven is a convex function.

Proof: Take two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$, let $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ for any $\lambda \in [0,1]$, then: $f(\lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y})) = f((\lambda x_1 + (1-\lambda) y_1), (\lambda x_2 + (1-\lambda) y_2)) = \lambda x_1 + (1-\lambda) y_1 - (\lambda x_2 + (1-\lambda) y_2) = \lambda x_1 + (1-\lambda) y_1 - \lambda x_2 - (1-\lambda) y_2 = \lambda(x_1 - x_2) + (1-\lambda)(y_1 - y_2) = \lambda f(x_1, x_2) + (1-\lambda) f(y_1, y_2) = \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y})$ ■

4.4 Convexity Analysis on Constraint Function

Known the first of constraint function $\bar{c}^T\boldsymbol{x} + n^T\boldsymbol{\gamma} - \delta_k^{(1)}\boldsymbol{o} + \delta_k^{(2)}\boldsymbol{p} \leq 0$, with $\boldsymbol{o}$ and $\boldsymbol{p}$ is a vector whose elements are one. It will be proven that the constraint is a convex function.

Proof: Take any $\boldsymbol{c}_A = (x_1, \gamma_1, \boldsymbol{o}_1, \boldsymbol{p}_1), \boldsymbol{c}_B = (x_2, \gamma_2, \boldsymbol{o}_2, \boldsymbol{p}_2)$ with $c_A, c_B \in \mathbb{R}^4$, and $\lambda \in [0,1]$: $\bar{c}^T(\lambda x_2 + (1-\lambda) x_1) + n^T(\lambda\gamma_2 + (1-\lambda)\gamma_1) - \delta_k^{(1)}(\lambda\boldsymbol{o}_2 + (1-\lambda)\boldsymbol{o}_1) + \delta_k^{(2)}(\lambda\boldsymbol{p}_2 + (1-\lambda)\boldsymbol{p}_1)$ $= \lambda\bar{c}^T x_2 + (1-\lambda)\bar{c}^T x_1 + \lambda n^T\gamma_2 + (1-\lambda) n^T\gamma_1 - \lambda\delta_k^{(1)}\boldsymbol{o}_2 - (1-\lambda)\delta_k^{(1)}\boldsymbol{o}_1 + \delta_k^{(2)}\boldsymbol{p}_2 + (1-\lambda)\delta_k^{(2)}\boldsymbol{p}_1 = \lambda\left(\bar{c}^T x_2 + n^T\gamma_2 - \delta_k^{(1)}\boldsymbol{o}_2 + \delta_k^{(2)}\boldsymbol{p}_2\right) + (1-\lambda)\left(\bar{c}^T x_1 + n^T\gamma_1 - \delta_k^{(1)}\boldsymbol{o}_1 + \delta_k^{(2)}\boldsymbol{p}_1\right)$. ■

Known the second of constraint function is $\bar{a}^T x + h^T\lambda - b\boldsymbol{o} \leq 0$ with $\boldsymbol{o}$ is a vector whose elements are all worth one. It will be proven that constraint is a convex function.

Proof: Take any $\boldsymbol{c}_C = (x_1, \lambda_1, \boldsymbol{o}_1), \boldsymbol{c}_D = (x_2, \lambda_2, \boldsymbol{o}_2)$ with $c_C, c_D \in \mathbb{R}^3$, for any $\tau \in [0,1]$: $\bar{a}^T(\tau x_2 + (1-\tau) x_1) + h^T(\tau\gamma_2 + (1-\tau)\gamma_1) - b(\tau\boldsymbol{o}_2 + (1-\tau)\boldsymbol{o}_1) = \tau\bar{a}^T x_2 + (1-\tau)\bar{a}^T x_1 + \tau h^T\lambda_2 + (1-\tau) n^T\lambda_1 - \tau b\boldsymbol{o}_2 - (1-\tau) b\boldsymbol{o}_1 = \tau(\bar{c}^T x_2 + n^T\lambda_2 - b\boldsymbol{o}_2) + (1-\tau)(\bar{c}^T x_1 + n^T\lambda_1 - b\boldsymbol{o}_1)$. ■

Known the third of constraint function is $\boldsymbol{N}^T\boldsymbol{\gamma} - \boldsymbol{P}^T\boldsymbol{x} = 0$. It will be proven that the constraint is an affine function.

Proof $\boldsymbol{N}^T\boldsymbol{\gamma} - \boldsymbol{P}^T\boldsymbol{x} = 0 \rightarrow [\boldsymbol{N} \quad -\boldsymbol{P}]^T \begin{bmatrix}\boldsymbol{\gamma}\\ \boldsymbol{x}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$, (33)

let $[\boldsymbol{N} \quad -\boldsymbol{P}]^T = \boldsymbol{A}$, $\begin{bmatrix}\boldsymbol{\gamma}\\ \boldsymbol{x}\end{bmatrix} = \boldsymbol{v}$, $\begin{bmatrix}0\\0\end{bmatrix} = o$ then (33) become

$$\boldsymbol{Av} = 0;\ \boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{v} \in \mathbb{R}^n, o \in \mathbb{R}^m ■ \quad (34)$$

Known the fourth of constraints function $\boldsymbol{H}^T\boldsymbol{\lambda} - \boldsymbol{Q}^T\boldsymbol{x} = 0$. The structure of this constraint as same as the form of third constraint, so the same way the constraint function is an affine function.

Known the fifth of constraint function $\boldsymbol{c}_1^T\boldsymbol{x} - \boldsymbol{c}_1^* = 0$. It will be proven that the constraint is an affine function.

Proof: let $\boldsymbol{C} = [\boldsymbol{c}_1^T]$ is a matrix with vectors entry $\boldsymbol{c}_1$, $\boldsymbol{y} = \boldsymbol{x}$, and $b = \boldsymbol{c}_1^*$, then $\boldsymbol{c}_1^T\boldsymbol{x} - \boldsymbol{c}_1^* = \boldsymbol{C}\boldsymbol{y} - b = 0$, with $\boldsymbol{C} \in \mathbb{R}^{m\times n}$, $\boldsymbol{y} \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. ■

4.5 Analysis based on the Karush-Kuhn-Tucker Method

In this section, the solution of the model formulation obtained (32) is determined using the KKT Method, and its Lagrangian functions are as follows:

$$\begin{aligned} &L\left(\boldsymbol{x},\boldsymbol{\gamma},\boldsymbol{\lambda},\delta_k^{(1)},\delta_k^{(2)},\mu,\nu,\sigma,\beta,\theta\right) = \delta_k^{(1)} - \delta_k^{(2)} \\ &+\mu\left(\bar{\boldsymbol{c}}^T\boldsymbol{x} + \boldsymbol{n}^T\boldsymbol{\gamma} - \delta_k^{(1)} + \delta_k^{(2)}\right) + \nu(\boldsymbol{N}^T\boldsymbol{\gamma} - \boldsymbol{P}^T\boldsymbol{x}) \\ &+\sigma(\bar{\boldsymbol{a}}^T\boldsymbol{x} + \boldsymbol{h}^T\boldsymbol{\lambda} - b) + \beta(\boldsymbol{H}^T\boldsymbol{\lambda} - \boldsymbol{Q}^T\boldsymbol{x}) + \textstyle\sum_{l=1}^{k-1}\theta(\boldsymbol{c}_l^T\boldsymbol{x} - \boldsymbol{c}_l^*), \end{aligned} \tag{35}$$

where $\mu,\nu,\sigma,\beta,\theta$ is a vector of Lagrange multipliers that correspond to each of the constraints. The conditions of the KKT consist of:

1. Stationary Conditions:

$$\begin{aligned} &\frac{\partial\mathcal{L}}{\partial\boldsymbol{x}} = 0, \Longrightarrow \textstyle\sum_{i=1}^{k-1}\theta_i\boldsymbol{c}_i + \mu\bar{\boldsymbol{c}} + \nu(-\boldsymbol{P}) + \sigma\bar{\boldsymbol{a}} + \beta(-\boldsymbol{Q}) = 0 \\ &\frac{\partial\mathcal{L}}{\partial\boldsymbol{\gamma}} = 0, \Rightarrow \boldsymbol{n}\mu + \boldsymbol{N}\nu = 0 \\ &\frac{\partial\mathcal{L}}{\partial\boldsymbol{\lambda}} = 0, \Rightarrow \boldsymbol{h}\sigma + \boldsymbol{H}\beta = 0 \\ &\frac{\partial\mathcal{L}}{\partial\delta_k^{(1)}} = 0, \Rightarrow 1 - \mu = 0 \Rightarrow \mu = 1 \\ &\frac{\partial\mathcal{L}}{\partial\delta_k^{(2)}} = 0, \Rightarrow -1 + \mu = 0 \Rightarrow \mu = 1 \end{aligned} \tag{36}$$

2. Feasibility Conditions: all of the constraint functions in (32).
3. Complementary Duality, and Non-negativity Conditions:

$$\begin{aligned} &\mu_i\left(\bar{\boldsymbol{c}}^T\boldsymbol{x} + \boldsymbol{n}^T\boldsymbol{\gamma} - \delta_k^{(1)} + \delta_k^{(2)}\right) = 0, \\ &\sigma_i(\bar{\boldsymbol{a}}^T\boldsymbol{x} + \boldsymbol{h}^T\boldsymbol{\lambda} - \boldsymbol{b}) = 0, i = 1,2,\dots,r. \\ &\mu \geq 0, \sigma \geq 0 \end{aligned} \tag{37}$$

By solving the equations system, the optimal solution is: $\boldsymbol{x} = \frac{c_l}{\bar{a}}$, $\mu = 1$, $\sigma = -\boldsymbol{a}$, $\nu = -\boldsymbol{N}^{-1}\boldsymbol{n}$, $\theta = -\left(\sum_{i=1}^{k-1}\boldsymbol{c}_i\right)\left(\bar{\boldsymbol{c}} + \boldsymbol{P}(\boldsymbol{N}^{-1}\boldsymbol{n})\right)$, $\boldsymbol{\gamma} = \frac{(\boldsymbol{N}^T)^{-1}\boldsymbol{P}^T\boldsymbol{c}_l}{\bar{\boldsymbol{a}}}$, $\boldsymbol{\lambda} = \frac{\boldsymbol{b}-\boldsymbol{c}_l}{\boldsymbol{h}}$, $\boldsymbol{\beta} = \boldsymbol{H}^{-1}\boldsymbol{h}\bar{\boldsymbol{a}}$, $\delta_k^{(1)} = \bar{\boldsymbol{c}}^T\left(\frac{\boldsymbol{c}_l}{\bar{\boldsymbol{a}}}\right) + \boldsymbol{n}^T(\boldsymbol{N}^T)^{-1}\boldsymbol{P}^T\left(\frac{\boldsymbol{c}_l}{\bar{\boldsymbol{a}}}\right)$, and $\delta_k^{(2)} = -\left(\bar{\boldsymbol{c}}^T\left(\frac{\boldsymbol{c}_l}{\bar{\boldsymbol{a}}}\right) + \boldsymbol{n}^T(\boldsymbol{N}^T)^{-1}\boldsymbol{P}^T\left(\frac{\boldsymbol{c}_l}{\bar{\boldsymbol{a}}}\right)\right)$. Because the solution to each variable does not depend on the other variables, the solution must be unique.

4.6. Discussion of Gap Analysis

Based on the results of SLR using the PRISMA method, four main clusters were found in the literature, namely "RO," "Multi-Objective Optimization," "Convex Optimization," and "Linear Programming," which showed a strong relationship between keywords. Although "Lexicographic Method" and "Optimization Model" are rarely found, topics such as "Multi-Objective Optimization" and "RO" are increasingly relevant, especially in the 2021-2022 period. Further analysis shows that there is a research gap related to the application of lexicographic methods in the linear robust optimization model, which has not been discussed in depth, especially with the use of polyhedral uncertainty sets. This study seeks to fill this gap by analyzing the existence and uniqueness of the solution through the KKT method and convex analysis, making a new contribution in the combination of lexicographic methods with a robust multi-objective linear optimization model. This research has the potential for novelty because it fills in the gaps in the related literature.

4.7. Discussion of Research Analysis

This discussion reviews in detail the application of the Lexicographic Method and the formulation of the RC model in the problem of multi-objective robust linear optimization, with a focus on the existence, singularity, and stability of the solution. The Lexicographic Method addresses multi-objective optimization by sorting objective functions by priority, completing them sequentially, and ensuring solutions that meet the constraints from highest to lowest priority. The RC formulation aims to address parameter uncertainty in optimization, ensuring the optimal solution remains valid under the worst-case scenario. The use of polyhedral uncertainty sets in RC models allows for more flexible and realistic representation of uncertainty, so that the resulting solution is more stable and efficient. Analysis of the existence and singularity of the solution using the KKT method and convex analysis guarantees that the optimal solution obtained is unique, globally optimized, and meets all constraints.

5. CONCLUSION

Based on SLR, the problem of robust multi-objective linear optimization models with the Lexicographic Method is an important novelty implemented in the field. The application of Lexicographic Methods in robust multi-objective linear optimization provides the ability to deal effectively with data uncertainty, through an objective-prioritizing approach. RC formulations with polyhedral uncertainty sets are able to accommodate data variations, providing a flexible and adaptive structure for various uncertainty conditions. The convexity study and application of the KKT method proves that the solution resulting from this model not only exists but is also unique, thus guaranteeing an optimal solution.

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