IDEALIZED STRESS-STRAIN RELATIONSHIP IN TENSION OF REINFORCE CONCRETE MEMBER FOR FINITE ELEMENT MODEL BASED ON HANSWILLE'S THEORY

Hardi Wibowo [1]

ABSTRACT

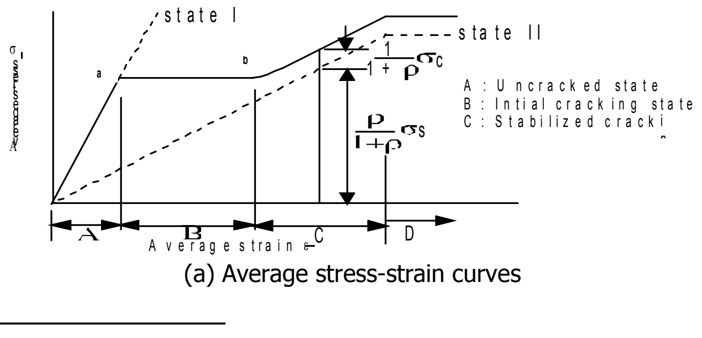
Untuk penganalisaan kontrol retak (crack control) pada model struktur beton bertulang (reinforced concrete/RC) menggunakan software finite element seperti DIANA diperlukan pemodelan hubungan tegangan-regangan (stress-strain relationship). Data model hubungan tegangan-regangan ini dalam rangka mengakomodasi efek rekatan (bond-slip effect) antara tulangan (reinforcement) dengan beton (concrete) yang melingkupinya. Hubungan tegangan-regangan yang diperlukan adalah nilai rata-rata tegangan-regangan pada tulangan dan beton. Data hubungan tegangan-regangan rata-rata ini bisa diturunkan dengan menggunakan teori Hanswille. Pada tulisan ini akan diuraikan mengenai teori Hanswille untuk menentukan hubungan tegangan-regangan tersebut dan diberikan satu contoh perhitungan dan curva hubungan tegangan-regangan rata-rata dari sebuah batang beton bertulang.

GENERAL

To model reinforced concrete member for finite element (FE) analysis, reinforcement steel bars were modeled as embedded elements. In this element, the bar elements do not have independent degrees of freedom. Instead, the stiffness of the bar elements were superposed on that of mother concrete elements. In FE model, perfect bonding between concrete and embedded reinforcement is assumed.

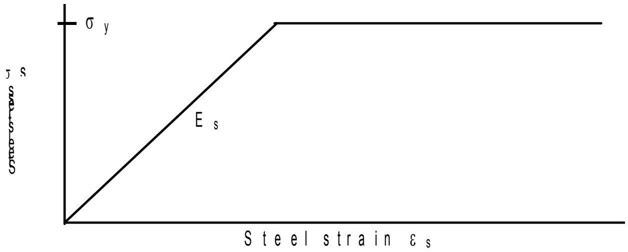
Bond-slip effect between reinforcement and surrounding concrete can be taken into account by using an average stress-strain relationship of reinforced concrete including tension stiffening effect. The average stress-strain relation derived from the bond-slip differential equation proposed by Hanswille will be explained here.

Fig.1 below shows the schematic figure for this stress-strain relationship. The state I corresponds to perfect bonding, while the state II to perfect cracking. The average stress $\overline{\sigma}$ of a RC member is expressed in terms of average steel stress $\overline{\sigma}_s$ and average concrete stress $\overline{\sigma}_c$,

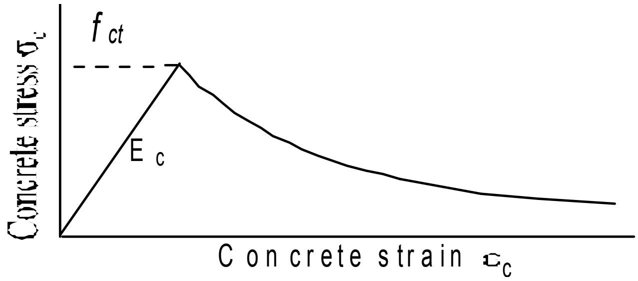


(a) Average stress-strain curves

[1] *Staf Pengajar Jurusan Teknik Sipil Fakultas Teknik Universitas Diponegoro*



(b) stress-strain curve for steel



(c) stress-strain curve for concrete

Fig.1 Schematic figure for Stress-strain curves of RC member in tension.

$$\bar{\sigma} = \frac{1}{V}\int_V \sigma dV = \frac{1}{1+\rho}\left(\bar{\sigma}_c + \rho\bar{\sigma}_s\right) \quad (1)$$

where for a uniaxial stress state,

$$\bar{\sigma}_c = \frac{1}{L}\int_0^L \sigma_c dV, \quad \bar{\sigma}_s = \frac{1}{L}\int_0^L \sigma_s dV \quad (2)$$

Where :

$V = A_t L$ is the total volume of the RC member with the cross-sectional area $A_t = A_c + A_s$ and length L.

$\rho = \frac{A_s}{A_c}$: reinforcement ratio

A_s=Cross-sectional area of reinforcement
A_c=Cross-sectional area of concrete

Under the assumption of perfect bonding used in the present smeared crack FE analysis, the average concrete strain equals the average steel strain. However, the average concrete strain $\bar{\varepsilon}_c$ includes contribution of crack opening in concrete. We can separate the average strain into an intact part and a cracking part, as

$$\bar{\varepsilon}_c = \frac{1}{L}\int_0^L \frac{du}{dx}dx = \frac{1}{L}\left(\int_{L^*} \frac{du}{dx}dx + \sum[u]\right) \quad (3)$$
$$= \varepsilon_{cm} + \sum w/L$$

where ε_{cm} denotes the average strain over the intact part L^*; $w = [u] \equiv u^+ - u^-$ is the crack width, and the summation is taken over all cracks in L. In summary, since normal stresses are zero in the cracking part, we have

$$\bar{\sigma}_c = \sigma_{cm}, \quad \bar{\sigma}_s = \sigma_{sm},$$
$$\bar{\varepsilon}_c = \bar{\varepsilon}_s = \varepsilon_{sm} = \varepsilon_{cm} + \Sigma w/L \quad (4)$$

where the overbar denotes averaged quantities over the total region, while the subscript "m" stands for averaged quantities over the intact part. In the present smeared FE analysis, we obtain the averaged quantities over the total region as output.

Since the elastic perfectly plastic model is assumed for stress-strain relation of reinforcement steel as shown in Fig.1(b), the stress-strain relation for concrete as shown in Fig.1(c) was derived from the average stress-strain relation from Hanswille's. The derived stress-strain relation of concrete is modeled as a multi-linear curve in FE analysis.

CONSITUTIVE MODEL AND BOND-SLIP DIFFERENTIAL EQUATION

Consider a differential length d_x of reinforcement embedded in concrete as shown in Fig.2, then the force due to bond between steel reinforcement and the surrounding concrete must be same as the change of axial force on steel or the concrete cross-section.

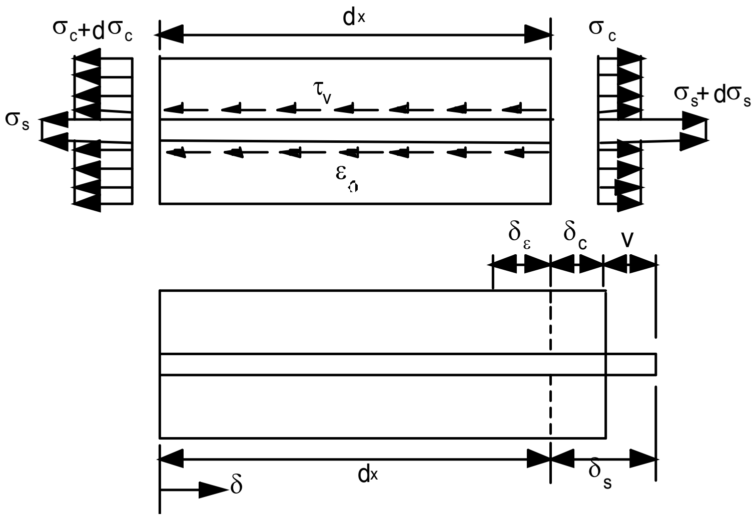


Fig.2 Stresses in differential element of RC member.

Considering equilibrium of forces in the longitudinal direction

$-d\sigma_c(x)A_c = d\sigma_s(x)A_s = \tau_v(x)U_sd_x$ .... (5)

Where :

τ_v = bond stress
σ_c = Stress in concrete
σ_s = Stress in concrete
U_s = Perimeter of cross-section of reinforcing bar $=\pi d_s$
d_s = Diameter of steel bar
x = Longitudinal coordinate of the member

Dividing Eq.5 by A_sd_x

$$\frac{d\sigma_s(x)}{d_x} = \tau_v(x)\frac{U_s}{A_s} = \frac{4}{d_s}\tau_v(x) \quad \text{...............(6)}$$

Assuming that the cross-section remains constant then the slip v or relative displacement between steel and concrete is equal to the difference of strain between steel and concrete. From Fig.2

$v = \delta_s-\delta_c-\delta_\varepsilon$ ....(7)

or...... $\frac{dv}{dx} = \varepsilon_s - \varepsilon_c - \varepsilon_0$ ....(8)

where :
v = relative displacement between steel and concrete
δ_c = displacement in concrete
δ_s = displacement in steel
δ_ε = displacement due to shrinkage
ε_0 = Shrinkage strain
ε_c = Strain in concrete
ε_s = Strain in steel

In Eq.8, ε_s and ε_c can be replaced by steel stress and concrete stress respectively and n=modular ratio= $\frac{E_s}{E_c}$

E_s = modulus of elasticity of steel
E_c = modulus of elasticity of concrete

$$\frac{dv}{dx}=\frac{1}{E_s}\left[\sigma_s(x)-n\sigma_c(x)\right]-\varepsilon_0 \text{ .... (9)}$$

Hanswille used the following constitutive model defined in exponential form for bond stress and relative displacement or slip

$$\tau_v(x) = Af_{cw}v^N(x) \text{ .... (10)}$$

where A and N are constants and are obtained empirically, and f_{cw}=compressive strength of cube of concrete. Using the value of $\tau_v(x)$, we get

$$\frac{d^2v}{dx^2}=\frac{4Af_{cw}}{d_s}\left(\frac{1+n\rho}{E_s}\right)v^N(x) \text{ .... (11)}$$

Now the solution of v=0 and $\frac{dv}{dx}=0$, will give the length of slip region v(x) :

$$\frac{dv}{dx}-\frac{2}{1-N}\left[\frac{(1-N)^2}{1+N}\times\frac{2(1+n\rho)}{E_s}\times\frac{Af_{cw}}{d_s}\right]^{\frac{1}{1-N}}\times x^{\frac{1+N}{1-N}}$$
....(12)

The boundary condition described above means that at a location where there is no slip between steel reinforcement and the concrete is taken as the origin of coordinate system.

FIRST CRACK

In a RC member subjected to axial force, when the stress attains the tensile strength of concrete, the first crack appears and then the stress of concrete and steel at the cracking region changes and a relative displacement (slip v) between steel and concrete is produced. The stress condition as shown in Fig.3 is produced. The length of the region where crack produces relative displacement is denoted here by L_{ER} and is called introduction length or transmission length.

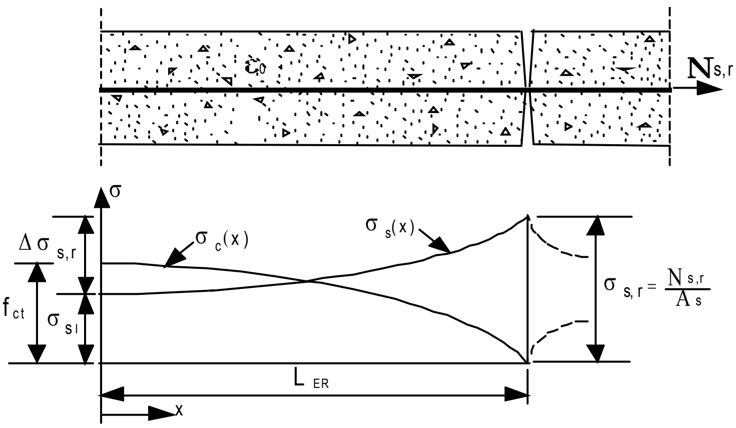


Fig.3 Stress of concrete and steel reinforcement after first crack in a RC member subjected to axial tensile force

To describe the stress $\sigma_c(x)$ and $\sigma_s(x)$ of concrete and steel in the region of length L_{ER}, these quantities are represented as a function of slip. The coordinate system chosen is shown in Fig.3. At the origin of this coordinate system, no slip is produced between steel and concrete, i.e. at x = 0, $\frac{dv}{dx} = 0$

From Eq.9, for x=0, we have

$$\frac{dv}{dx} = \frac{1}{E_S}\left[\sigma_S(0) - n\sigma_C(0)\right] - \varepsilon_0 = 0 \quad \text{......... (13)}$$

$\sigma_s(x)$ at $x=L_{ER}$ is $\sigma_{s,r}$ and $\sigma_s(x)$ at x=0 is $\sigma_{s,I}$. When at $x=L_{ER}$, $\sigma_s(x)$ is $\sigma_{s,r}$ then $\sigma_c(0)=f_{ct}$. Equating the total force at the crack and at the section x=0

$\sigma_{s,r}A_s = A_c\sigma_c + A_s\sigma_s$ or $\sigma_{s,r}\times\rho = \sigma_c + \rho\sigma_s$

At x = 0, $\rho\times\sigma_{s,r} = \sigma_c(0) + \rho\sigma_s(0)$

And then we find

$$\sigma_{S,r} = \frac{f_{ct}(1+n\rho)}{\rho} + E_S\varepsilon_0 \quad \text{.................... (14)}$$

$$\Delta\sigma_{S,r} = \frac{\sigma_{S,r} - E_S\varepsilon_0}{1+n\rho} \Rightarrow 1+n\rho = \frac{\sigma_{S,r} - E_S\varepsilon_0}{\Delta\sigma_{S,r}} \quad \text{.... (15)}$$

$$\Delta\sigma_{S,r} = \frac{E_S}{1+n\rho} \times \frac{dv}{dx} \quad \text{............................. (16)}$$

INITIAL CRACKING STATE

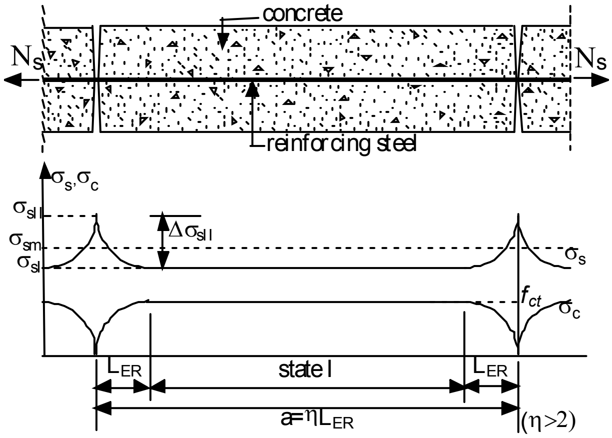
In a reinforced concrete member subjected to axial tensile force, when the stress attains the tensile strength of concrete, the first crack appears and the relative slip between steel reinforcement and surrounding concrete is produced. In this method for crack width evaluation, the same constitutive relation between the bond stress τ_v and the slip v was used as used by Hanswille's theory, given in Eq.10. In this equation, N is not a non-dimensional parameter but dimensional one. If the unit of length is cm, then Hanswille reported that A=0.58 and N=0.3 are standard values for a deformed bar.

After first cracking of concrete, further increase of the axial force increases the number of cracks and accordingly deformation due to cracking, and thus spacing between adjacent cracks reduces. Hence, depending upon the spacing between the cracks, two cracking states can be defined, one is initial cracking state and the second one is stabilized cracking state.

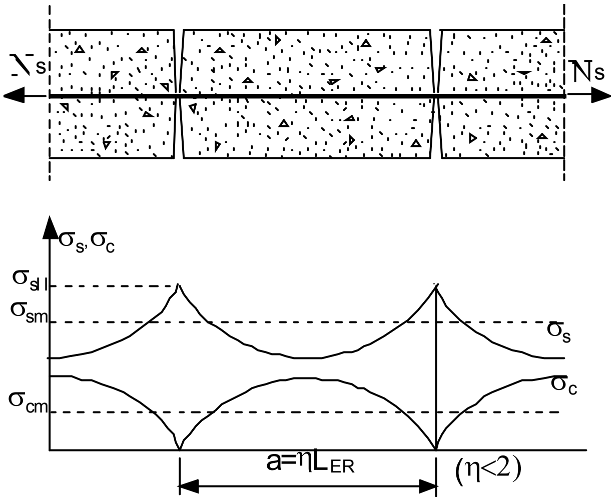
In the initial cracking state as shown in Fig.4(a), the transmission length of two adjacent cracks do not overlap each other and there is a state I region between two adjacent cracks. Since there is no relative slip in the state-I region, there is no interaction between two adjacent cracks, i.e. the opening of one crack does not affect the width of around cracks.

Further increase of axial force causes the generation of more cracks till the spacing between cracks become so small so that the transmission lengths of cracks overlap each other. This is called stabilized cracking state as shown in Fig.4(b). In this state, the opening of one crack affects the widths of around cracks. Crack width expression in both cracking states is given by w=2v at the crack position.

The crack width w_R for the first crack is derived as given by Eq.18 for $\varepsilon_{S,I} < \bar{\varepsilon} < \varepsilon_{is}$ with $\sigma_{S,r}$ and $\Delta\sigma_{S,r}$ are given in Eq.14 and $\Delta\sigma_{S,r} = \frac{f_{ct}}{\rho}$ respectively.



(a) Initial cracking stage



(b)Stabilized cracking stage
Fig.4: Stress distributions in cracked steel reinforced concrete member.

ε_{is}=Average strain at the boundary between initial cracking state and stabilized cracking state. ε_{is} is obtained from the average steel strain $\varepsilon_{s,m}$ in the initial cracking state. We have

$$\varepsilon_{s,m} = \frac{1-N}{\eta} \Delta\varepsilon_{s,II} \alpha + \varepsilon_{s,I} \quad \text{................ (17)}$$

where α is defined as

$$\alpha = \left(\frac{\sigma_{s,II} - \varepsilon_0 E_s}{\sigma_{s,r} - \varepsilon_0 E_s} \right)^{\frac{1-N}{1+N}} \quad \text{.......................(18)}$$

and

$$\Delta\varepsilon_{s,II} = \varepsilon_{s,II} - \varepsilon_{s,I}$$

At the boundary of initial and stabilized cracking states, α=1
thus $\sigma_{s,II}=\sigma_{s,r}$ and $\Delta\varepsilon_{s,II}=\Delta\varepsilon_{s,r}=\varepsilon_{s,II}-\varepsilon_{s,I}$

η is a non-dimensional factor defining crack spacing in terms of transmission length.

Hanswille has considered the possibility of stress drop due to further cracking in the initial cracking state and thus he has also proposed expression for crack width other than first crack, thus giving crack width less than first crack. As long as initial cracking state prevails, the crack width expression for first crack gives the maximum crack width. This procedure is intended to evaluate the maximum crack width. Thus, in our proposed method, we assume a constant stress state in the initial cracking state as shown in Fig.1.

STABILIZED CRACKING STAGE

In the stabilized cracking stage, we have:

$$\alpha = \left(\frac{\sigma_{sII} - \varepsilon_0 E_s}{\sigma_{sr} - \varepsilon_0 E_s}\right)^{\frac{1-N}{1+N}} \qquad (19)$$

and η=a/L_{ER} is the non dimensional crack spacing. Hanswille proposes the following expression as a mean value of η_m on the basis of the statistical experimental data.

$$\eta_m = \eta_{\max}/1.1 = \frac{2}{1.1}(\alpha^* - \beta^*) \qquad (20)$$

where

$$\alpha^* = \left(\frac{\sigma_{sII} - E_s\varepsilon_0}{\sigma_{sr} - E_s\varepsilon_0}\right)^{\frac{1-N}{1+N}\frac{2+N}{2}}$$
$$\beta^* = \left(\frac{\sigma_{sII} - E_s\varepsilon_0}{\sigma_{sr} - E_s\varepsilon_0} - 1\right)^{\frac{1-N}{1+N}\frac{2+N}{2}} \qquad (21)$$

When evaluating the crack width, it is needed to know σ_{sII} from FE analysis. σ_{sII} is the stresses in steel in state II as defined in Fig. 4(b). The relationship between σ_{sII} and the average stresses in FE analysis can be derived from the equilibrium condition.

N_c=Axial force of RC slab
At uncracked location, the force will be shared by concrete and steel reinforcement.

$$N_c = A_t \cdot \bar{\sigma}$$
$$N_c = A_c \cdot \bar{\sigma}_c + A_s \cdot \bar{\sigma}_s \qquad (22)$$

At crack, the force is taken by only reinforcing steel, and the resulting stresses on the reinforcing bars is σ_{sII}.

$$\sigma_{sII} = \frac{N_c}{A_s} \qquad (23)$$

Putting the value of N_c from Eq.2.40 into Eq.2.41 we get the relation

$$\sigma_{sII} = \frac{A_c}{A_s}\bar{\sigma}_c + \bar{\sigma}_s \qquad (24)$$

Or...... $\sigma_{sII} = \frac{1}{\rho}\bar{\sigma}_c + \bar{\sigma}_s$ .... (25)

Thus the Eq.24 can be used for getting the value of σ_{sII} from $\bar{\sigma}_c$ and $\bar{\sigma}_s$. The $\bar{\sigma}_c$ and $\bar{\sigma}_s$ are obtained from out put of FEM analysis. The average strain derived from the bond-slip differential is given by Hanswille, as follows:

$$\varepsilon_{sm} = \frac{\sigma_{sII}}{E_s}\left[1 - \frac{\Delta\sigma_{sII}}{\sigma_{sII}}\left[1 - \frac{1-N}{\eta_m}\alpha\left[1 - \left[1 - \frac{1}{2}\frac{\eta_m}{\alpha}\right]^{\frac{2}{1-N}}\right]\right]\right] \qquad (26)$$

From this average strain expression, we can specify the average stress-strain of a RC member, which is used in the FE analysis.

AN EXAMPLE

Finally, we have conclusion that stress-strain relationship to model RC member for FEM has three stages i.e.: un-cracked stage, initial cracking stage, and stabilized cracking stage (see fig.1a). In the un-cracked stage (line 0-a), range of strain is $0 \le \bar{\varepsilon} \le \varepsilon_{sI}$ and average stress is $\bar{\sigma} = \frac{f_{ct}(1+n\rho)}{1+\rho}$. In the initial cracking state (line a-b) range of average strain is $\varepsilon_{sI} \le \bar{\varepsilon} \le \varepsilon_{is}$ where ε_{is} is average strain at the boundary between initial cracking state and stabilized cracking state. In this point α=1 and η=2, then. $\varepsilon_{is} = \frac{f_{ct}}{E_S}\left(\frac{1-N}{2\rho}+n\right)+\varepsilon_0$ and average stress is $\bar{\sigma} = \frac{\sigma_{S,r}\,\rho}{1+\rho}$. And in the Stabilized cracking state range of average strain is $\varepsilon_{is} \le \bar{\varepsilon}$ $\bar{\varepsilon} = \varepsilon_{sm}$ and equations below are hold:

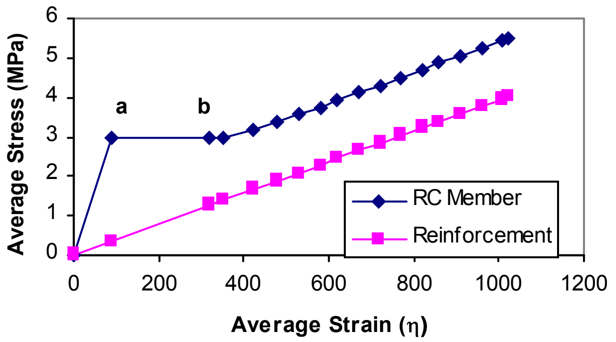
$$\varepsilon_{sm} = \frac{\sigma_{sII}}{E_S}\left[1-\frac{\Delta\sigma_{sII}}{\sigma_{sII}}\left[1-\frac{1-N}{\eta_m}\alpha\left[1-\left[1-\frac{1}{2}\frac{\eta_m}{\alpha}\right]^{\frac{2}{1-N}}\right]\right]\right]$$

In this condition $\sigma_C = 0$ therefore $\bar{\sigma} = \frac{\sigma_{S,II}\,\rho}{1+\rho}$ and $\sigma_S = \sigma_{s,II}$, where $\sigma_{S,r} \le \sigma_{S,II} \le$ Yield strength of reinforcement.

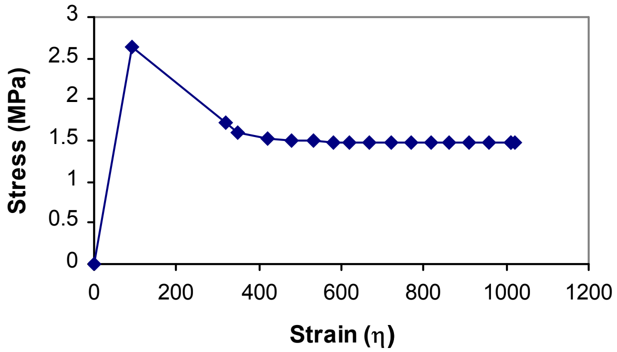
Following is an example of calculation using data as below: Modulus of elasticity of steel **E** = 210000 N/mm^2, Compressive strength of concrete = $\boldsymbol{\sigma_{cy}}$ = 40 N/mm^2 , Tensile strength of concrete $\mathbf{f_{ct}}$ = 2.69 N/mm^2, Reinforcement ratio ρ = 0.0191, Modular ratio $\mathbf{n_o}$ = 7, Bond slip constant **A** = 0.291 (in mm), Bond slip constant **N** = 0.3, Shrinkage strain of concrete $\boldsymbol{\varepsilon_o}$ = 0. The calculation result can be presented as a table and figures as shown below:

Ave. strain	**Ave.RC stress (Mpa)**	**Ave.reinf. stress (Mpa)**	**Note**
0,00000	0,00000	0,00000	Point 0
0,00009	2,99250	0,35291	Point A
0,00032	2,99250	1,27677	Point B
0,00035	2,99872	1,39261	
0,00042	3,18614	1,65887	
0,00048	3,37356	1,87031	
0,00053	3,56099	2,07046	
0,00058	3,74841	2,26533	Stabilized
0,00062	3,93583	2,45719	cracking
0,00067	4,12325	2,64715	state
0,00072	4,31067	2,83583	
0,00077	4,49809	3,02361	
0,00082	4,68551	3,21075	
0,00086	4,87293	3,39741	
0,00091	5,06035	3,58372	
0,00096	5,24777	3,76975	
0,00101	5,43519	3,95558	
0,00101	5,45393	3,97416	
0,00102	5,51015	4,02987	

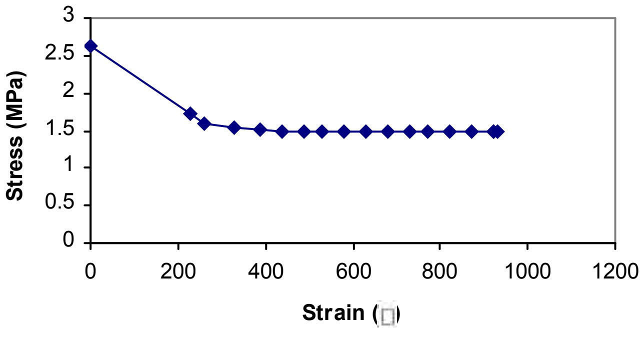
Table 1.: Calculation result



(a). Stress-strain relationship for RC member and reinforcement



(b). Stress-strain relationship for concrete



(c). Multi-linear tension softening curve for DIANA input data

Fig. 5. Result of calculation of stress-strain relationship and Multi-linear tension softening curve

REFERENCES

AASHTO, 1998, LRFD *Bridge Specifications, SI units, Second edition, American Association of State Highway and Transportation Officials*, 444 North Capitol Street, N.W., Suite 249 Washington, D.C. 20001

Clark, A. P., 1956, Cracking in reinforced concrete flexural members, *ACI Journal, Proceedings,* Vol.52, No.8.

Crisfield, M. A. and Wills, J.,1989, Analysis of R/C panels using different concrete models, *Journal of Engineering Mechanics*, ASCE, Vol. 115, No. 3

Hanswille, G., 1986, *Zur Rißbreitenbeschränkung bei Verbundträgern, Technisch-Weissenschaftliche Mitteilungen*, Institut für Konstruktiven Ingenieurbau Ruhr-Universität Bochum, Mittelilung Nr. 86-1.

Holmberg, A. and Lindgren, S., 1970, *Crack spacing and crack width due to normal force or bending moment*, Document D2, 1970, National Swedish building research.

JSCE, 1998, Committee of Concrete, *Standard specifications for concrete structures* [Design], (in Japanese).

azaqpur, A. G. and Nofal, M., 1990, Analytical modelling of non-linear behaviour of composite bridges, *Journal of Structural Engineering*, ASCE, Vol. 116, No. 6.