# AN ELLIPTIC ELASTIC-PLASTIC ASPERITY MICRO-CONTACT MODEL

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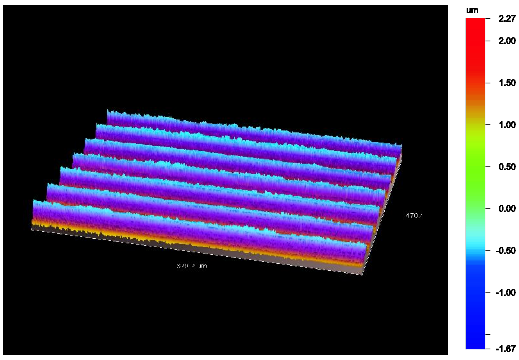
***Abstract***

*A theoretical model for the elastic-plastic contact of ellipsoid bodies is presented in this paper. Contact parameters, such as the mean contact pressure, the contact area and the contact load as a function of the contact interference are formulated and modeled in the three different contact regimes: elastic, elastic-plastic and fully plastic. The model introduces a new simple method to analyze the complexity of the elliptical integral by an accurate approximation so that the calculation can be executed very fast.*

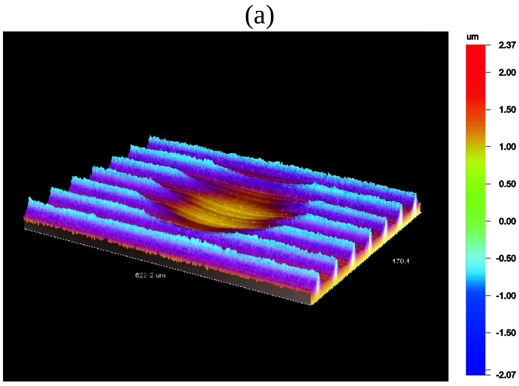
*Key words: contact mechanics, elliptic, elastic-plastic, asperity.*

## INTRODUCTION

The surface textures of most of the engineering surfaces are oriented with the direction of the relative motion of cutting tools to the workpieces, see Fig. 1. Different processing methods will produce different asperity radii of curvature, and therefore different ellipticity ratios of the micro-contacts are formed. The profile of the asperities generally contains various curvatures for various directions.




(a)




(b)

**Figure 1. Elliptical contact of asperities: a) before contact and b) after high load of spherical indenter is removed.**

In accordance to these facts, several models have been proposed to extend the isotropic asperity contact model into an anisotropic asperity contact model. Horng [1], for instance, extended the *CEB* [2] model and Jeng and Wang [3] extended the *ZMC* [4] model for the elliptical contact situation.

Even though the results of McCool [5] showed that for anisotropy rough surfaces with a random distribution of asperity radii differ negligibly from those of the isotropic one, for the deterministic contact situation the ellipticity of the contact cannot be simplified, especially when studying the change of the micro-geometry. In this thesis the micro-geometric change of the surface after unloading is the main topic. Therefore, the elliptical contact situation will be considered for the analysis.

The present study offers a new accurate elliptic elastic-plastic asperity contact model. An elastic-perfectly plastic with no-strain hardening effect material behavior is considered. It is shown that in the fully plastic contact regime the mean contact pressure is lower than the indentation hardness as often reported and the transition from the elastic-plastic to the fully plastic state is material dependent.

## ELLIPTIC ELASTIC CONTACT

Figure 2 shows the general situation of the contact between two elastic bodies. Eccentricity of the ellipse $e$ is defined as:

$$e^2 = 1 - \left(\frac{a}{b}\right)^2, \qquad a < b \tag{1}$$

where $a$ and $b$ denote the semi-minor and semi-major radius of the elliptic contact area. The mean effective radius $R_m$ is represented as:

$$\frac{1}{R_m} = \frac{1}{R_x} + \frac{1}{R_y} = \frac{1}{R_{x1}} + \frac{1}{R_{x2}} + \frac{1}{R_{y1}} + \frac{1}{R_{y2}} \tag{2}$$

where $R_x$ and $R_y$ are the effective radii of curvature in principal $x$ and $y$ direction; subscripts 1 and 2 indicate body 1 and body 2, respectively.

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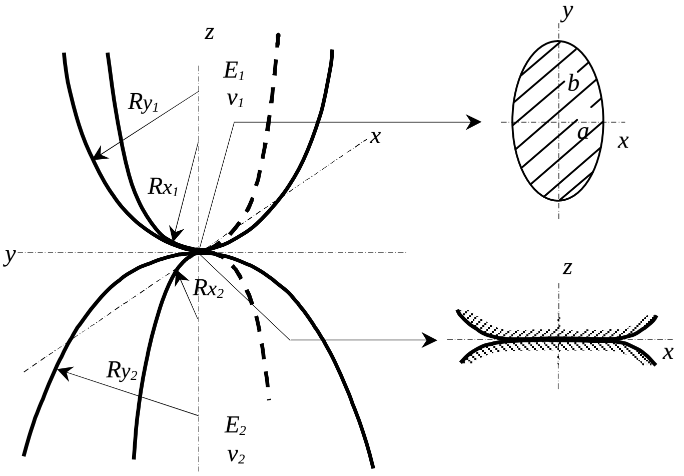


**Figure 2. Geometry of contraformal contact..**

From the theory of elasticity, the maximum contact pressure $p_o$, the semi-major contact ellipse radius $b$ and the interference of an asperity $\omega$ can be written as [6]:

$$p_0 = \frac{3}{2}p = \frac{3P}{2\pi ab} \tag{3}$$

$$b = \left[\frac{3\mathbf{E}(e)PR_m}{\pi E(1-e^2)}\right]^{1/3} \tag{4}$$

$$\omega = \frac{2\mathbf{K}(e)}{\pi}\left[\frac{\pi(1-e^2)}{4\mathbf{E}(e)R_m}\right]^{1/3}\left(\frac{3P}{4E}\right)^{2/3} \tag{5}$$

where

$$\left(\frac{a}{b}\right)^2 = \frac{\mathbf{E}(e)}{\mathbf{K}(e) + \dfrac{R_x}{R_y}\left[\mathbf{K}(e) - \mathbf{E}(e)\right]} \tag{6}$$

and

$$\frac{1}{E} = \frac{1-{v_1}^2}{E_1} + \frac{1-{v_2}^2}{E_2} \tag{7}$$

$p$ is the mean contact pressure, $P$ is the contact load, $a$ is the semi-minor radius of the elliptic contact area, $E$ is the effective elastic modulus and $R_x$ and $R_y$ are the effective radii of curvature as defined in Eq. (2). $\mathbf{K}(e)$ and $\mathbf{E}(e)$ are the complete elliptic integrals of the first and second kind, respectively:

$$\mathbf{K}(e) = \int_0^{\pi/2}(1-e^2\sin^2\varphi)^{-0.5}d\varphi \tag{8}$$

$$\mathbf{E}(e) = \int_0^{\pi/2}(1-e^2\sin^2\varphi)^{0.5}d\varphi \tag{9}$$

The elastic contact area $A_e$ and the contact load $P_e$ can be expressed in terms of the contact interference $\omega$ by combining Eqs. (3), (4) and (5):

$$A_e = \left[\frac{\mathbf{E}(e)}{\mathbf{K}(e)(1-e^2)^{0.5}}\right]2\pi R_m\omega \tag{10}$$

$$P_e = \left[\frac{\pi\mathbf{E}(e)^{0.5}}{2\mathbf{K}(e)^{1.5}(1-e^2)^{0.5}}\right]\frac{4\sqrt{2}}{3}E{R_m}^{0.5}\omega^{1.5} \tag{11}$$

Eqs. (10) and (11) contain elliptical integrals, whose values must be found from tables. Approximations for the complete elliptic integrals have been introduced, for example, by Reussner [7].

The semi-minor radius of the contact area $a$, the semi-major radius of the contact area $b$ and the elastic interference $\omega$ are defined as [8]:

$$a = \alpha\left(\frac{3PR_m}{2E}\right)^{1/3} \tag{12}$$

$$b = \beta\left(\frac{3PR_m}{2E}\right)^{1/3} \tag{13}$$

$$\omega = \gamma\left(\frac{9P^2}{32E^2R_m}\right)^{1/3} \tag{14}$$

where $\alpha$, $\beta$ and $\gamma$, are

$$\alpha = \kappa^{1/3}\left[\frac{2}{\pi}\mathbf{E}(m)\right]^{1/3} \tag{15}$$

$$\beta = \kappa^{-2/3}\left[\frac{2}{\pi}\mathbf{E}(m)\right]^{1/3} \tag{16}$$

$$\gamma = \kappa^{2/3}\left[\frac{2}{\pi}\mathbf{E}(m)\right]^{-1/3}\frac{2}{\pi}\mathbf{K}(m) \tag{17}$$

In Eqs. (15)-(17) $\kappa$, $\mathbf{E}(m)$ and $\mathbf{K}(m)$ are defined by:

$$\kappa \approx \left[1 + \sqrt{\frac{\ln(16/\lambda)}{2\lambda}} - \sqrt{\ln 4} + 0.16\ln\lambda\right]^{-1}$$
$$for\ \ 0 < \lambda \le 1 \tag{18}$$

$$\mathbf{E}(m) \approx \frac{\pi}{2}(1-m)\left[1 + \frac{2m}{\pi(1-m)} - 0.125\ln(1-m)\right] \tag{19}$$

$$\mathbf{K}(m) \approx \frac{\pi}{2}(1-m) + \left[\frac{\pi}{2}(1-m)\right] \times$$
$$\left[\frac{2m}{\pi(1-m)}\ln\left(\frac{4}{\sqrt{1-m}}\right) - 0.375\ln(1-m)\right] \quad (20)$$

where $m = 1 - \kappa^2$ and the curvature ratio $\lambda$

$$\lambda = \frac{R_x}{R_y} \quad (21)$$

Combining Eqs. (12), (13) and (14), the contact area and the contact load can be presented in terms of $\omega$ as:

$$A_e = 2\pi R_m \omega \frac{\alpha\beta}{\gamma} \quad (22)$$

$$P_e = \frac{4\sqrt{2}}{3} E R_m^{\ 0.5}\left(\frac{\omega}{\gamma}\right)^{1.5} \quad (23)$$

The mean contact pressure $p_e$ is simply expressed in terms of $\omega$ by dividing Eq. (23) by Eq. (22):

$$p_e = \frac{2}{3} p_m = \frac{2\sqrt{2}}{3\pi}\frac{E}{\alpha\beta}\left(\frac{\omega}{R_m\gamma}\right)^{0.5} \quad (24)$$

Substituting $p_m = K_v H$ into Eq. (24) yields the critical interference $\omega_1$:

$$\omega_1 = \frac{\pi^2}{2}\alpha^2\beta^2\gamma R_m \frac{K_v^{\ 2} H^2}{E^2} \quad (25)$$

where $K_v$ is a hardness coefficient related to the Poisson's ratio $v$. Recently, [9] have derived $K_v$ based on the von Mises shear strain energy criterion as:

$$K_v = 0.4645 + 0.3141v + 0.1943v^2 \quad (26)$$

**ELLIPTIC FULLY PLASTIC CONTACT**

Based on the experimental results of a deformable sphere in contact with a hard flat in the fully plastic contact regime [10], the mean contact pressure is not equal to the hardness [11] but lower. For a more general representation the mean contact pressure in the fully plastic regime can be related to the hardness as:

$$p_p = c_h H \quad (27)$$

where $c_h$ is the hardness coefficient for the fully plastic contact regime.

It was shown [10] that in the fully plastic contact regime the contact area is simply a truncation of the undeformed asperity geometry as was postulated by [12]. For the elliptical contact situation this can be expressed as:

$$\omega = \frac{a^2}{2R_x} = \frac{b^2}{2R_y} \quad (28)$$

Rearranging and substituting Eq. (28) into $A = \pi ab$ yields the contact area in the fully plastic $A_p$:

$$A_p = 2\pi\sqrt{R_x R_y}\,\omega \quad (29)$$

The contact load $P_p$ is equal to the contact area multiplied by the mean contact pressure, or

$$P_p = 2\pi\omega\sqrt{R_x R_y}\,c_h H \quad (30)$$

The solid expression for the onset of fully plastic interference $\omega_2$ is not known, therefore it is estimated. A simple analysis is done based on the contact load. At $\omega = \omega_2$, the contact load is equal to Eq. (60). At the same time, the contact load had it been elastic as would be equal to Eq. (23). Therefore, the following inequality can be established:

$$\frac{4\sqrt{2}}{3} E R_m^{\ 0.5}\left(\frac{\omega_2}{\gamma}\right)^{1.5} > 2\pi\omega_2\sqrt{R_x R_y}\,c_h H \quad (31)$$

Eq. (31) can be rewritten as:

$$\omega_2 > \left(\frac{\pi^2}{2}\alpha^2\beta^2\gamma\frac{R_m}{E^2}K_v^{\ 2}H^2\right)\left(\frac{3}{2}\frac{\gamma}{\alpha\beta}\frac{\sqrt{R_x R_y}}{R_m}\frac{c_h}{K_v}\right)^2 \quad (32)$$

Substituting Eq. (15) to (21) and Eq. (25) into Eq. (32) yields:

$$\omega_2 > \left(\frac{3\kappa}{2}\frac{\mathbf{K}(1-\kappa^2)}{\mathbf{E}(1-\kappa^2)}\frac{1+\lambda}{\sqrt{\lambda}}\frac{c_h}{K_v}\right)^2\omega_1 \quad (33)$$

The minimum value of $\omega_2$ may also be further estimated using experimental results. The fully plastic regime of a half-space indented by a rigid sphere according to Francis [13], starts at $A/A_c$ = 113.2 and according to Johnson [6] full plasticity starts at $Ea/YR \approx 40$ or $P/P_c \approx 360$. Based on the experimental results of Chaudhri *et al.* [14] and Tabor [5] for the contact problem of a deformable sphere and a rigid flat, the fully plastic contact regime (as indicated by a constant mean contact pressure) starts at $A/A_c \approx 40$ for phosphor-bronze, 90 for brass and 160 for steel. Or in general:

$$\frac{A}{A_c} = c_A \quad (34)$$

Substituting Eqs. (22) and (29) into Eq. (34) and rearranging results:

$$\frac{\omega_2}{\omega_1} = c_A \frac{R_m}{\sqrt{R_x R_y}} \frac{\alpha\beta}{\gamma} \qquad (35)$$

Using Eqs. (15) - (21) and substituting into Eq. (35) yields:

$$\frac{\omega_2}{\omega_1} = c_A \frac{\sqrt{\lambda}}{1+\lambda} \frac{1}{\kappa} \frac{\mathbf{E}(1-\kappa^2)}{\mathbf{K}(1-\kappa^2)} \qquad (36)$$

Fig. 3 shows the plot of the onset of fully plastic contact as a function of the ellipticity ratio $\lambda$ based on Eq. (3.63). It can be seen that for the range of $\lambda$ of 0.55 to 1, $\omega_2/\omega_1$ is almost constant at a value of about 45 for $c_A$ = 90.

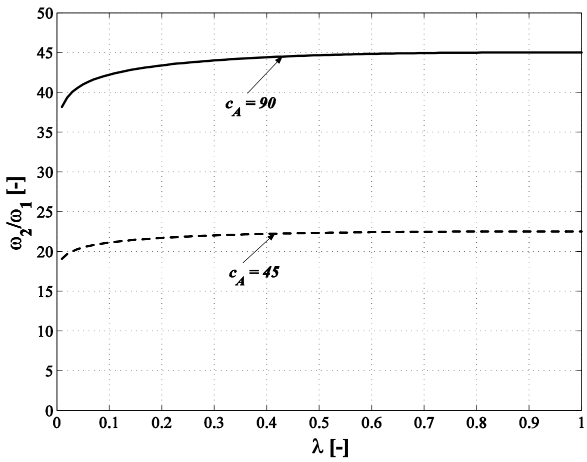


**Figure 3. Onset of full plasticity as a function of the ellipticity ratio $\lambda$ of Equation (36).**

## ELLIPTIC ELASTIC-PLASTIC CONTACT

Based on a statistical analysis of spherical indentations, Francis [13] presented the dependence of the mean contact pressure $p_{ep}$ on $\omega$ for the elastic-plastic contact situation, which may be analogously characterized by a logarithmic function, as follows:

$$p_{ep} = a_1 + a_2 \ln\left(\frac{\omega}{r}\right) \qquad (37)$$

where $a_1$ and $a_2$ are two constants to be determined, and $r$ is the circular contact radius. Since the area of elliptical contact $A = \pi ab$, the elliptical contact radius can be represented as $(ab)^{1/2}$. Substituting this relation into Eq. (37) results:

$$p_{ep} = a_1 + a_2 \ln\left(\frac{\omega}{\sqrt{ab}}\right) \qquad (38)$$

At the critical interference $\omega_1$, Eq. (38) can be expressed in the form:

$$\frac{2}{3} K_v H = a_1 + a_2 \ln\left(\frac{\omega_1}{\sqrt{ab}}\right) \qquad (39)$$

and at the inception of full plasticity $\omega_2$:

$$c_h H = a_1 + a_2 \ln\left(\frac{\omega_2}{\sqrt{ab}}\right) \qquad (40)$$

By simultaneously solving Eqs. (39) and (40), the parameters $a_1$ and $a_2$ can be determined in terms of the properties of the contact:

$$a_1 = c_h H - \frac{H\left(c_h - \frac{2}{3}K_v\right)}{\ln\omega_2 - \ln\omega_1} \ln\left(\frac{\omega_2}{\sqrt{ab}}\right) \qquad (41)$$

$$a_2 = \frac{H\left(c_h - \frac{2}{3}K_v\right)}{\ln\omega_2 - \ln\omega_1} \qquad (42)$$

Substituting Eqs. (41) and (42) into (37) gives the mean contact pressure in the elastic-plastic deformation region as follows:

$$p_{ep} = c_h H - H\left(c_h - \frac{2}{3}K_v\right) \frac{\ln\omega_2 - \ln\omega}{\ln\omega_2 - \ln\omega_1} \qquad (43)$$

Zhao *et al.* [4] proposed a relation of the elastic-plastic contact area as a function of $\omega$. The relation was modeled by using a polynomial expression to join the expression of the contact area at $\omega = \omega_1$ and $\omega = \omega_2$ smoothly. A 'template' cubic polynomial function is defined as:

$$y = 3x^2 - 2x^3 \qquad (44)$$

By this function all four boundary conditions: $A_{ep} = A_e$, $dA_{ep}/d\omega = dA_e/d\omega$ at $\omega = \omega_1$ and $A_{ep} = A_p$, $dA_{ep}/d\omega = dA_p/d\omega$ at $\omega = \omega_2$ are satisfied. The transformation involves translating and scaling $\omega$ so that $\omega = \omega_1$ and $\omega = \omega_2$ correspond to $x = 0$ and $x = 1$, respectively, where:

$$x = \frac{\omega - \omega_1}{\omega_2 - \omega_1} \qquad (45)$$

The expression of $A_{ep}$ after scaling is:

$$A_{ep} = A_e + (A_p - A_e)\left[3\left(\frac{\omega-\omega_1}{\omega_2-\omega_1}\right)^2 - 2\left(\frac{\omega-\omega_1}{\omega_2-\omega_1}\right)^3\right] \qquad (46)$$

Substituting Eqs. (22) and (29) into Eq. (46) gives the relation between the elastic-plastic contact area $A_{ep}$ in terms of contact interference $\omega$ as:

$$A_{ep} = 2\pi R_m \omega \frac{\alpha\beta}{\gamma} + \left( 2\pi \sqrt{R_x R_y}\,\omega - 2\pi R_m \omega \frac{\alpha\beta}{\gamma} \right) \left[ 3\left( \frac{\omega - \omega_1}{\omega_2 - \omega_1} \right)^2 - 2\left( \frac{\omega - \omega_1}{\omega_2 - \omega_1} \right)^3 \right] \quad (47)$$

The elastic-plastic contact load $P_{ep}$ is defined using Eqs. (43) and (47), $P_{ep} = p_{ep}A_{ep}$, as:

$$P_{ep} = A_{ep}\left[ c_h H - H\left( c_h - \frac{2}{3}K_v \right) \frac{\ln\omega_2 - \ln\omega}{\ln\omega_2 - \ln\omega_1} \right] \quad (48)$$

## CONCLUSIONS

A theoretical model for the normal contact of elastic-plastic of ellipsoid bodies has been presented. In order to predict the contact behaviour formulae describing the contact parameters have been developed.

The model introduces a new simple method to analyze the complexity of the elliptical contact situation so that the cost of the calculation time can be reduced dramatically.

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