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GEOMETRIC CONSTRAINTS AND TANGENTIAL AND RADIAL STRESSES OF ROTATING DISCS WITH VARIABLE THICKNESS

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ABSTRACT

This article presents an algorithm along with Scilab codes to calculate radial and tangential stresses in rotating disc with varying thickness. The continuous change of thickness is represented by stepped; discrete ones.The geometry of disc is represented by two arrays, each of whose elements represents the radius and the thickness respectively. The stresses arise due to combination of rotation and prescribed pressures at the outer and inner radiuses of the disc. The stresses are limited to linearly elastic cases; loadings and rotation causing either non-linear elastic or plastic strains are beyond the scope of this work. Linear inequality onstraints on the values of the radiuses are prescribed in order to prevent overlapping of radiuses, and thus maintain the validity of the disc geometry. Codes are designated for Scilab for ease of use.

Keywords: *Rotating discs, Optimization, Mechanical Design, Geometric Constraints, Turbine Discs*

1. INTRODUCTION

Rotating discs have numerous applications in current engineering practice, in particular in advanced system and critical systems. Probably, the most popular and numerous applications are in the turbomachinery in which the blades and impeller, which are a kind of rotating disc, are the most critical part in those machines. Loadings received by these discs are combination of mechanical and termal; the former can be further classified into those loadings from rotation and pressures.

The pressure at the inner surface of a disc arises from the connection between shaft and the disc. The shaft can be either an individual part connected to the disc, or an integral part of it. This work is limited to the former one. Each of the configurations will dictate unique pressure and stresses at the connection point.

This article presents the work to model the geometry of stepped discs with a pair of arrays representing radiuses of the steps and thicknesses of ring segments respectively. These arrays is used to calculate the radial and tangential stresses at the interfaces due to loadings of rotation and pressures. The case in this work assumes that the strains are in the linearly elastic region, and thus allow the development to use Hooke's law.

2. ROTATING DISCS

The geometry of a disc is an annular ring with a hole in the center, and the section of the disc has a shape of a stepped rectangle. The hole has a radius R_m, and facilitates a connection with a shaft. The outermost radius is designated by R_2 whose index is chosen to follow that in Seireg's book in order to conform to the original labeling. The index of the radius runs from 2 to m. Thicknesses of steps are designated by L_m with index runs from 2 to m, i.e. the same as that of the radius. The material properties of the disks relevant to this article are density and Poisson's ratio. The operating conditions are described by rotating speed, pressure at the outermost radius, and that at the innermost radius. The last one originates from the fit between the disc and the shaft. In this article, the pressure at the innermost radius must facilitate a tight a fit capable of transmitting torque without slip.

3. STRESSES AT THE STEPS

The expressions for radiuses l and tangential stresses at the steps of the disc follows, and are duplicated from, the formulation by Seireg [3] and Timoshenko [4]. The purpose of presenting the formulation of the stresses is to provide the sequence of derivations and to show the equations employed in the Scilab scripts.

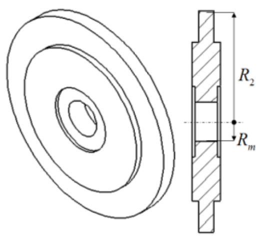

Figure 1 Stepped Discs

Based on the range on index for the radius explained previously, the radiusesl and tangential stresses at the step (a.k.a. interface) can be expressed as the following equations respectively.

$$(\sigma_r)_{n+1} = -\frac{1}{2}\left[1 + \left(\frac{L_n}{L_{n+1}}\right) P_{n+1}\right]$$

$$(\sigma_t)_{n+1} = -B_n\left(\frac{L_{n+1}}{L_n}\right)P_n + \left[E_n + \frac{B_n}{2} - \frac{\nu}{2}\left(\frac{L_n}{L_{n+1}}\right)\right]P_{n+1} + F_n V^2$$

for $n \in [2, m-2]$, *m* is the innermost surface of the disc, and where

$$E_n = \frac{R_{n+1}^2}{R_n^2 - R_{n+1}^2}$$

$$V = \omega R_2$$

$$B_n = \frac{2R_n^2}{R_n^2 - R_{n+1}^2}$$

$$F_n = \frac{(3+\nu)\rho}{4}\left(\frac{R_n}{R_2}\right)^2 + \frac{(1-\nu)\rho}{4}\left(\frac{R_{n+1}}{R_2}\right)^2$$

$$P_{n+2} = K_n V^2 - Q_n P_n + U_n P_{n+1}$$

and where

$$K_n = \frac{A_n}{C_n}$$

$$Q_n = \frac{B_n L_{n-1}}{C_n L_n}$$

$$U_n = \frac{D_n}{C_n}$$

$$A_n = \frac{(3+\nu)\rho}{4R_2^2}\left(R_n^2 - R_{n+1}^2\right)$$

$$C_n = \frac{2R_{n+2}^2}{R_{n+1}^2 - R_{n+2}^2}$$

$$D_n = \frac{(1-\nu) + (1+\nu)R_n^2}{R_n^2 - R_{n+1}^2} + \left(\frac{L_n}{L_n}\right)\left(\frac{(1+\nu) + (1-\nu)R_{n+1}^2}{R_{n+1}^2 - R_{n+2}^2}\right)$$

Besides geometric dimensions, properties, and operating conditions, all of the equations above use artificial variables for clarity and readability purposes.

The equation for P_{n+2} deserves a special explanation. For n = 2, the equation needs P_3 to be known, but it is not known yet. The workaround of this contradiction is by running this equation twice, each of which uses a uniquely assumed P_3; the assumptions are completely random. Since this equation is linear in P's, the resulting P_m's will enjoy linear proportion with the true values of P_3 and P_m respectively. This proportion allows the use of interpolation to calculate the true P_3. Then, this equation for P is repeated for the final calculation for the remaining stresses at the steps.

4. OBJECTIVES OF OPTIMIZATION

This section present brief introduction to the optimization of rotating disc design. Seireg proposes eight expressions for the objective of rotating disc design, and all of them are functions of combinations of radiusesl and tangential stresses respectively, volume, and inertia. The combination is of partial ones. The objectives belows are listed in Seireg's book, and is partially presented here.

- Minimize the maximum tangential stress.
- Minimize the average tangential stress.
- Minimize the difference between the maximum and minimum tangential stresses.
- Minimize the maximum equivalent stress.
- Maximise the inertia over volume

The list above demonstrates that importance of radiusesl and tangential stresses as an intermediate variables that will form the objective functions most suitable for the current design at hand. The list is far from exhausted, but this article will not discuss more objective functions for relevance purposes. The central idea in this article is focused on representation of geometry, calculation of stresses, and constraint among the geometric variables. Jahed investigated rotating stepped discs under thermal stress, and reported the optimization to minimize the weight [1]. . Alexandrova reported an investigation of constant thickness disc under plane state of stress in elastic-perfectly plastic isotropic regime [2].

5. ALGORITHM AND CODES FOR STRESS CALCULATION

The algorithm to calculate the stresses are straightforward, and is derived from all of equations mentioned previously. The followings present the algorithm and the Scilab codes to calculate the stresses.

Algorithm

- Declare global variables for the followings: (1) properties, i.e. density in kg per meter cube and Poisson's ratio, (2) operating conditions, i.e. rotation speed in rpm, pressure at the inner and outer radiuses respectively in Pascal, and (3) gravitation in meter per second square.
- Calculate intermediate variables A, B, C, and D, as expressed in previous equations.
- Calculate intermediate variables K, Q, and U, as expressed in previous equations.
- Assume a couple of interface pressures at the outermost interface; these assumed pressures must be different in order to allow interpolation.
- Using the pressures, calculate two values of internal pressure, each of which is based on each assumed pressure as mentioned above.
- Interpolate the actual pressure at the outermost interface using the values of internal pressure computed above and the actual value of the internal pressure.
- Using the computer pressure of the outermost interface, calculate the pressures at the remaining interface.
- Calculate the tangential and radiuses. stresses respectively which develop at the interface.

The codes for Scilab macros are shown below

Codes

```
global nu, g, rho, omega, Pin, Pout, numring
g = 9.81; // meter per second square
nu = 0.3;
rho = 7860; // kg per meter cubic
omega = 1500; // rpm
Pout = 0;
```

Codes

```
Pin = 0;
numring = 4;

R = x(1:numring+1); R = [R(1) R]
L = x(numring+2:length(x)); L = [L(1) L]

m = length(R)-2; // substraction of 2 for indexing of loops

// the followings are arbitrary assumption for linear interpolation
P31 = -121;
P32 = 42.2;

// calculating A, B, C, D
A = (R(2:m)/R(2))^2 - (R(3:m+1)/R(2))^2;
A = A*(3+nu)*rho/g/4;

B = 2*(R(2:m)./R(3:m+1))^2/((R(2:m)./R(3:m+1))^2-1);

C = 2 ./ ((R(3:m+1)./R(4:m+2))^2-1);

D1 = (R(2:m)/R(3:m+1))^2 ./((R(2:m)./R(3:m+1))^2-1);
D1 = (1-nu)/(1+nu)*D1;
D2 = (L(2:m)./L(3:m+1)).*((1+nu+(R(3:m+1)./R(4:m+2))^2*(1-nu)) ./((R(3:m+1)./R(4:m+2))^2-1));
D = D1 + D2;

// K, Q, and U
K = A ./ C;
Q = B ./ C .* (L(1:m-1)./L(2:m));
U = D ./ C;

// Calculating Pin
// start with creating a couple vectors of pressure,
// the outer pressures are assign the actual outer pressure
// the subsequent pressures, i.e. index = 3, are assigned arbitrary values
P1 = zeros(1,length(R)); P1(2)=Pout; P1(3)=P31;
P2 = zeros(1,length(R)); P2(2)=Pout; P2(3)=P32;
// the next step is populating the vectors of pressure
for i = 2 : m // m has been replaced by m-2 in the previous line
P1(i+2)=K(i-1)-Q(i-1)*P1(i)+U(i-1)*P1(i+1);
P2(i+2)=K(i-1)-Q(i-1)*P2(i)+U(i-1)*P2(i+1);
end
```

Line **Codes**

```
// finally, the P(3) is obtained by interpolation of Pin
// index m must be added by 2 in order to compensate the shifted index
P1(3) = P1(3) + (Pin-P1(m+2))*(P2(3)-P1(3))/(P2(m+2)-P1(m+2));

// Recalculate Pressure distribution using new P1(3)
for i = 2 : m
P1(i+2)=K(i-1)-Q(i-1)*P1(i)+U(i-1)*P1(i+1);
end

// Now, P1 is the pressure distribution.
// Its P1(2) and the last element are P_out and P_in respectively

E = 1 ./ ((R(2:m)./R(3:m+1))^2 - 1);
F = ((R(2:m)/R(2))^2)*(3+nu)*rho/4/g + ((R(3:m+1)/R(2))^2)*(1-nu)*rho/4/g;

sigmaR = -0.5 * (1+(L(2:m)./L(3:m+1))).*P1(3:m+1);
sigmaR = [Pout sigmaR Pin];

// stresses
sigmaT = -B .* (L(3:m+1)./L(2:m)).*P1(2:m) + (E + 0.5*B - (L(2:m)./L(3:m+1))).*P1(3:m+1) + F;
sigmaT = [0 sigmaT 0];
```

The Scilab codes above takes arrays of radiuses and thicknesses respectively as inputs. With constant parameters specified as global variables, the codes will return arrays of radiusesl and tangential stresses respectively at all interfaces. The radiusesl stresses at the outer and inner radiuses are exactly the loading pressures at those radiuses respectively. The tangential stresses at the same place are zero.

6. CONSTRAINTS AND BOUNDS FOR GEOMETRIC VARIABLES

The geometric variables of the discs that constitute optimization variables are the interface radiuses and thicknesses, i.e. R_i for $i \in [3,m-1]$ and L_j for $j \in [2,m-1]$. Each of these variables will normally have upper and lower bounds. The handling of the bound during optimization is taken care of by the optimization internal bound-handling routines, and therefore is beyond the scope of this article.

The radius variables needs constraint to prevent overlapping values that violate the geometric relation between them. In verbal statements, the constraint needs to maintain that the subsequent or adjacent radius in the inner side of the discs is smaller a given particular radius. This constraint can be realized by a linear inequality constraints as the followings.

$$\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} R_2 \\ R_3 \\ \vdots \\ R_m \end{Bmatrix} \geq \begin{Bmatrix} \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_m \end{Bmatrix}$$

where each ε_i is a minimum allowable distance between two adjacent radiuses.

7. CASE OF STRESS CALCULATION

A disc is made of four rings whose diameters are 200 mm, 100 mm, 75 mm, 50 mm, and 25 mm, respectively. The ring thicknesses are 12,5 mm, 41,7 mm, 70,8 mm, and 10 mm, respectively. Figure 2 shows the disc. The density is

7360 kg per meter cube, and the disc operates at 10 000 rpm. No pressure loadings are prescibed at outer and inner rings. The stresses at the interface are calculated using the codes mentioned previously, and the results are listed in Table.1.

The results indicate the tensile nature of radial stresses and the compressive nature of the tangential stresses. Outer and inner surfaces have no stress as specified by the condition of operations. Interfaces experience a combined stresses loadings from the radial and tangential ones.

Table 1. Dimension and Stresses

No	***R*** **[mm]**	***L*** **[mm]**	σ_r **[Mpa]**	σ_t **[Mpa]**
2	200	12,5	0	0
3	100	41,7	0	-30,50
4	75	70,1	6,71	-26,25
5	50	100	17,08	-0,69
6	25		0	0

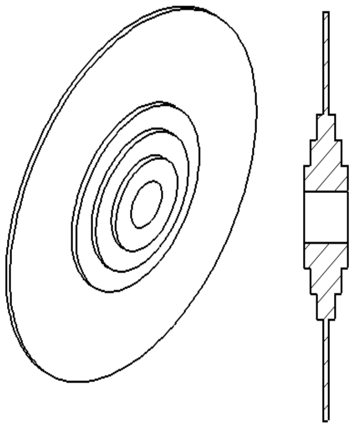

Figure 2. Tested Disc: 4 Segments

8. CONCLUSION

The codes have been tested and the results indicate that the codes consistently produce accurate stress values, bug free, and stable. Linear inequality constraints are always represented correctly, and thus provide guarantee of integrity of the radiuses in the disc. The results altogether point the readiness of the codes to be used in subsequent computation to design rotating discs.

9. REFERENCES

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